A Simulative Approach for Evaluating Electoral Systems

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Abstract This paper argues that simulation may be very useful to address some basic problems concerning the choice of the electoral system. A case study with ten electoral systems is analyzed as an example. The utility of including power indices is discussed in an appendix. A simulation program is briefly described.

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1. Introduction

At first sight, the problem of choosing the best electoral system for a Parliament cannot be solved. Some theorems – Arrow’s and McKelvey’s *in primis* – exclude the very possibility of finding out the optimal rule. In addition, there are too many variables involved: it may be too difficult to balance all of them. And finally, the adoption of complex methods appears at odds with the necessity of adopting a rule sufficiently easy to be understood and managed by voters. Improvements too technical or too cumbersome (or both) are unlikely to command the interest of policymakers.

These arguments are not compelling. It is true that it is impossible to find
out the optimal rule, but no theorem prohibits finding out an empirical criterion to establish whether a rule is better or worse than another. This is considered absolutely normal in economic policing matters: why not in the electoral ones? It is also true that the electoral system affects a lot of dimensions, but there is a large consensus that some are more important, and it makes sense to privilege them. Finally, quite complex rules may be managed and understood by voters. The panoply of methods actually adopted is very large, and the reduction of the choice set to them rules out the objection.

To sum up, scholars may suggest a way to choose the best electoral system. The result will obviously depend on the assumptions, that must be clear and reasonable, and, mostly, they will be cogent only for those who accept them. But this is true for all the results of normative social sciences, and there is no reason to discuss further this point with reference to our topic.

Actually, the main problem for the scholar who aims to suggest a criterion to optimally choose the electoral system lies elsewhere, i.e. in the availability of data to evaluate her/his results. In order to assess how a system performs vis-à-vis another one, one needs to compare the performance of the two systems with reference to the same set of real preferences of the voters. The best, and nearly the only, proxy available is the actual voting behavior. This is a very good proxy for many purposes, but not for the one we are dealing with here. The reason should be obvious: not only the voting behavior, but also preferences are affected by the electoral system actually in use. In UK and in USA, for instance, a rational elector will not vote for a minor party (bar obviously regional ones in UK), as it is bound to lose: s/he will choose her/his second (maybe further) best.

This obvious feature creates a vicious circle. Minor parties will not appear realistic (nor will they have access to a fair way to communicate their programs); hence they will not be perceived as a serious possibility. This rules out a second possible source of data: surveys. A (say) somehow-more-left-than-labor elector will not indicate as her/his first preference the British Left Labour Party, simply because it does not exist, or if it exists it is perceived as ‘weird’ – and probably it is, as the very fact that it is bound to lose in every election will probably select a number of weird people as its political entrepreneurs. Note that there are serious hints that in both USA and UK the demand for plausible political alternatives is large. Abstention suggests that many (actually, often most) voters are dissatisfied with the actual supply. More noticeably, a lot of parties take part in the elections (more than seventy in the last one in UK). Most of them are usually relegated to the dimension of quantum foam, but their very existence shows that the potential demand for parties is high, and that the supply would probably expand were the political market more contendible. Hence, data based on revealed preferences (from
elections or surveys) are of almost no use to compare electoral systems. With the possible exception of mixed-member systems, there is little way to obtain a useful answer to the question 'How would you vote were the electoral system X?' in a country where the system is Y.

However, useful answers may be produced by virtual subjects. It is difficult to imagine a field where the simulative approach may be more effective than in the assessment of electoral systems. There are two reasons for that; both should be quite obvious. The first is that the real world feature that must be simulated is very simple – a set of preferences. The assessment of the relative performance of electoral systems requires a set of preferences, but is entirely downstream of the reasons that produced a system or another one. In order to run her/his simulation, the experimenter must create a set of virtual electors. This set may be more or less realistic, and consequently more or less useful for normative suggestions; but once it has been created, it defines a case that has no differences from a real one. A 'virtual' case of a society that uses perfect proportionality and where there are some major parties and a cohort of minor ones provides nearly the same information offered by an analogous real-world case, like pre-reform Italy. The limits are only those of the accuracy of the simulation program, and we will return to this.

The second reason is analogous, and, perhaps, more important. While the virtual set of preferences is nearly as informative as a real one, the single virtual subject is identical to a real one. If we take for serious the requirements of the basic theorems of choice, Arrow's and May's, no preference must be privileged. Hence, there is no reason to ask why a given subject provided a given choice. If Lord Astor votes for the Communists, and if Mark Cyst, the dangerous troublemaker imagined by Parkinson (1962) prefers instead Lord Astor, these are their matters: the entire process of evaluating the result of their (and others') combined choices is again downstream. Hence, in this field (and this is a rare, if not unique, case) the virtual subjects include all the relevant features of the real ones.

In this paper we will argue in favor of the simulative approach for the evaluation of electoral systems. In Section 2 we will present some simulative programs, emphasizing ours. Section 3 shows that simulation may help in choosing the electoral system. Some concluding remarks are in Section 4. An appendix argues for the possibility of including power measures in simulation programs. Two other appendices deal with technicalities.

2. The simulation programs available

Given the utility and the versatility of the simulative approach for the analysis
of electoral systems, it is quite surprising that it is so little employed in the political science literature. There are some, but not that many, suggestive case studies, but very few papers address the matter we are dealing with here – to compare electoral systems. Navarra and Sobbrio (2001) use simulation to assess the motivations of the electoral reform in Italy. Bilodeau (1999) estimates the effect of the adoption of alternative vote in Canada. Bender and Haas (1996) analyze the contendibility of a two-party system. Brichta (1991) examines a specific mixed-member suggestion in a specific case, Israel. Valenzuela and Siavelis (1991) use opinion polls to assess the proportionality of Chilean electoral system. Finally, Lomborg (1997) uses simulation to find out equilibria in multiparty spatial models. Moving to the comparison of systems, there are some pioneering papers (see Mueller, 1989; Merrill, 1984 and 1985), but if we exclude those of our group, then only two in more or less recent years remain: Gambarelli and Biella (1992), who analyze the effect in Italy of a change to a number of electoral systems, and Christensen (2003), who compares six majoritarian systems, but without reference to a Parliament. Consequently, it is not surprising that the simulation programs so far available (like those developed by Accuratedemocracy) are of limited use for purposes of the kind suggested here (www.accuratedemocracy.org).

The simulations produced in this paper have been carried out with a specific program, called Governability and Representativeness (G&R – release 1.26), created with Microsoft Visual Basic 6.0; it simulates concrete voting situations using different electoral systems in an hypothetical country with 100 constituencies.1

The constituencies are characterized by their position on the axis of the electoral space (left-right) represented by a real number \( c \) in the interval \([-1,1]\), where \( c = -1 \) indicates an extreme left constituency, \( c = 1 \) an extreme right constituency and \( c = 0 \) a center constituency. The location of the positions on the axis for constituencies can be selected by the user or by the program, with a sequence of random numbers.2

The parties that participate in the electoral competition are characterized by their position on the same axis. Correspondingly, -1 means an extreme left party, +1 an extreme right party, 0 a center party.

The voters are characterized by their profiles. A profile for a voter is an

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1 Release 1.26 is the upgrade of 1.25 written by Trinchero (1998) with Microsoft Visual Basic 3.0; the main additions are the introduction of additional electoral systems and of improved indices for representativeness and governability. See Monella (2002) for details.

2 This feature is designed to allow the user to take into account the (possible) spatial concentration of the voters of a party. In the following examples we accepted the default (random) values.
ordering of its political preferences, expressed by a sequence of hexadecimal numbers. This allows using a single character (the program can work up to the maximum of 16 parties) and simplifies the elaboration of the preference orderings. For instance, with 12 parties, a profile can be represented by the sequence of preferences 567B9A832410, were the voter assigns 12 points (=B) to the fourth party, 11 (=A) to the sixth party and so on, eventually giving 1 point (=0) to the twelfth party. The parties are ordered according to their position on the left-right axis, so the first party is the leftmost party and the twelfth is the rightmost party. To each profile we associate a position \( e \) of the voter on the axis, computed as the weighted sum of the positions of parties on the axis, with weight 1 for the first preference (favorite party), 1/2 for the second, 1/3 for the third and so on. If necessary \( e \) is truncated at \(-1\) if it is smaller, or at \(+1\) if it is larger. The weights may be modified, to give more weight to left, or right, or center profiles.

The relative incidence \( y \) of each profile of preferences is transformed in the effective percentage in the constituency \( w \) by the product \( ce \), according to the following rules:

- if \( ce > 0 \), i.e. if both the profile and the constituency are either left or right, then \( w = y/1-ce \);
- if \( ce < 0 \), i.e. the profile and the constituency are on opposite sides, then \( w = y(1+ce) \).

The meaning of the rules is the following. If \( c \) and \( e \) are coherent, i.e. if a profile is leftist (rightist) in a leftist (rightist) constituency, its weight is increased; if not, the weight is reduced. The aim of this feature is to allow for 'countries' with geographical opinion clusters. Note, however, that it is possible to neutralize this procedure by assuming \( c = 0 \).

The weight of a profile is subsequently transformed into the relative weight of that profile in the constituency dividing it by the sum of the weights of all the profiles.

The output of the program is constituted by the schema of the parliament with the parties and their seats and by the index of representativeness (see Section 3). It is possible, selecting the preferred parties from a list, to determine a majority; the program will then compute the index of governability (see Section 3). Moreover, the program has a sample selection module and can generate the type of voters starting from a database.

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3 The hexadecimal system uses sixteen symbols \( 0123456789ABCDEF \) to represent the integers from 0 to 15.
This program provides interesting results (see Ortona, 1998), yet it has some limits, quite serious. First, it requires a database to work with; albeit the input may be duly modified, the result is still somehow casual, and may escape the requirements of the experimenter. Second, it does not include some complex, but relevant systems, like multi-district proportional and single transferable vote.4

3. The choice of the optimal electoral system

The choice of the best electoral system affects a lot of facets of the political process. For instance, and in addition to our main topics, representativeness and governability, we can mention: corruption (Myerson, 1993 and 2001; Persson, Tabellini and Trebbi, 2001); public spending (Persson and Tabellini, 1998 and 2001; Milesi-Ferretti et al., 2000); the overall welfare of a country (Mueller and Stratmann, 2000); the information and the participation of voters (Mudambi, Navarra and Nicosia, 1995; Mudambi, Navarra and Sobbrio, 1999); the responsiveness of the government’s choice to the preferences of the voters (Shugart, 2001); the power of the lobbies (Myerson, 1995); the incentives for politicians (Myerson, 1995; Riker, 1982); the possibility of strategic choices (Levin and Nalebuff, 1995); the complexity of the voting system (Levin and Nalebuff, 1995); the protection of the minorities (Levin and Nalebuff, 1995; Rae, 1995; Sen, 1995); the risk of extreme choices (Levin and Nalebuff, 1995); the use of the vote as a ‘voice’ device (Brennan and Hamlin, 1998; Sen, 1995). Others may be added.

Fortunately, there is a general agreement that the efficiency in representing electors’ will (representativeness, \( R \)) and the effect on the efficiency of the resulting government (governability, \( G \)) are of paramount relevance.5 There are at least two good reasons to privilege \( R \) and \( G \). First, to summon the representatives and to form a government are the basic aims of a Parliament (bar, obviously, to make laws). Possible pitfalls of other dimensions may be managed in other moments of the political process, but this is not the case for representativeness and governability, if we admit the sovereignty of the voters.

4 To take into account these problems, a completely new program (ALEX3) has been written, but it was not yet available when we run the simulations for this paper, so we are not able to provide results obtained from it. For a complete description, see Bissey, Carini and Ortona (2004).

5 A more detailed characterization of both \( R \) and \( G \) and of the related trade-off (\( R \) is likely to increase with the number of parties and \( G \) to decrease) is provided in Appendix B, through the definition of the indices employed to assess them. For a broader discussion, see Ortona (1998) and Bissey, Carini and Ortona (2004).
in choosing their representatives and that of the representatives in choosing the government. In addition, it is sensible to think that other dimensions are lexicographic with respect to them (bar possibly one, to be considered in next section).\footnote{Note however that the method outlined here may be extended to further dimensions, provided that suitable indices are available.} If this is so, the results obtained with reference to $R$ and $G$ will keep their validity irrespective of the dimensions judged relevant.

$R$ and $G$ may be evaluated through the assessment of plausible (albeit arbitrary) numerical indicators. The ones used in our simulations are briefly described in Appendix B (for further details, see Bissey, Carini and Ortona, 2004). We will label them $r$ and $g$ respectively.\footnote{A slightly different version of these indices has been employed also in Ortona (1998).} The range of both is the interval $0 - 1$.

Results for different electoral systems, referring to a single case, may be graphed, as in Figure 1. There are three possibilities. First, a system may be located north-east of all the others, like the system labeled with '?'. We define this system \textit{dominant}, and it is obviously the best one, as it has the highest value for both indices. Unfortunately, this system is very likely not to exist, given the trade-off between the two dimensions – hence the choice of the symbol '?'. Second, a system may be located south-west of at least another one, like system 4 with reference to system 2 (and to system '?', if it exists). We define such a system \textit{dominated}, and it may safely be excluded: no need to consider system 4, if system 2, better on both dimensions, is available. Third, systems may be neither dominant nor dominated, i.e. all of them are...
Pareto optimal, like 1, 2 and 3 in the graph (if '?' does not exist), and like (usually) plurality voting and proportional representation in real world. These are labeled alternative systems. Obviously, the rule we look for is useful only if it allows choosing among alternative systems. Note that there may be at most one (strongly) dominant system, while the dominated systems can be more than one.

**Example 1** Consider the following electoral systems: Pure Proportional \( [P] \), Threshold proportional \( (n\% \text{ clause}) \) \( [P-n] \), Prized Proportional (premium of \( n \) seats) \( [P(n)] \), Relative majority (First Past The Post) \( [M] \), Two-round run-off \( [2R] \), Condorcet method \( [C] \), Borda count \( [B] \), Approval voting \( [A] \), Mixed-member system \( (n\% \text{ } P \text{ } \text{and} \text{ } 100- \text{ n\% } M ) \) \( [I-n] \). Let the seats assignments to parties \( P_i \) to \( P_{12} \) be as in Table 1. Note that we assume that the proportional system assignment coincides with the distribution of voters \( v \).

Using the simulation program described in Section 2 and consequently the indices described in Appendix B, we obtain the Figure 2 and Table 2. There are six non-dominated systems \( P, P-4, P(20), M, 2R, A \). We may also exclude one system between \( M \) and \( 2R \), as weakly dominated by the other one.

In principle, to compare different electoral systems, we need voting results

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\( ^8 \) Here and below, without the exclusion of votes used in the proportional share from plurality voting.

\( ^9 \) See Appendix C for a sketchy description of these electoral systems.
for different systems: a majoritarian vote, a proportional vote, a list of voters’ ordered preferences for Condorcet voting, and so on. As we saw, it is usually impossible to collect these data from real world. But given a set of virtual electors, each with her/his preferences, it is perfectly possible to produce them. Given the votes, every system considered will produce a potential Parliament, and each Parliament will have a pair of values of \( r \) and \( g \). If a system will result as dominant, it is the good one; but, as we noticed, this result is very unlikely, given the trade-off between the two dimensions. What can be safely done is to rule out dominated systems, like system 4 in Figure 1. This operation will probably reduce considerably the number of potential candidates. Suppose that they reduce to three alternative systems, like in Figure 1 if system '?' does not exist. Apparently, what we need to compare them is a social utility function \( SUF \) – admittedly a quite formidable requirement, to say little. Actually, we may be satisfied with something less.

Let us admit the \( SUF \) for representativeness and governability to be a
typical Cobb-Douglas function in $g$ and $r$, $U = Kg^a r^b$, where $K$ is a suitable constant. We choose this form not only for its simplicity and versatility, but also for the meaning of $a$ and $b$, the partial elasticity of $U$ with reference to $g$ and $r$, respectively. Now consider two systems, $X$ and $Y$. We may write that:

$$X > Y \iff Kg_X^a r_X^b > Kg_Y^a r_Y^b$$

(1)

where $X > Y$ means that system $X$ is preferred to system $Y$.

Let $p = a/b$, and hence $a = pb$, so (1) reduces to:

$$X > Y \iff \left(\frac{g_X}{g_Y}\right)^p > \left(\frac{r_Y}{r_X}\right)^b$$

hence the condition may be written as:

$$p \ln \frac{g_X}{g_Y} > \ln \frac{r_Y}{r_X}$$

or

$$p > \frac{\ln \frac{r_Y}{r_X}}{\ln \frac{g_X}{g_Y}} \text{ if } g_X > g_Y$$

(2a)

$$p < \frac{\ln \frac{r_Y}{r_X}}{\ln \frac{g_X}{g_Y}} \text{ if } g_X < g_Y$$

(2b)

If $g_X = g_Y$ then the choice is made according to $r$.

**Remark 1** Note that the right hand sides of both equations (2a) and (2b) are positive if the two systems $X$ and $Y$ are both (strongly) non-dominated. In fact, $r_Y < r_X \Rightarrow g_Y > g_X$ and $r_Y > r_X \Rightarrow g_Y < g_X$.

**Remark 2** The ratio $p$ may also be characterized in another, more suggestive way. $p$ is the price in terms of a relative decrease of $r$ that the community accepts to pay for a given relative increase of $g$. If for instance we have
\( p = 2 \), it is worthwhile to accept a 20% reduction of \( r \) to gain a 10% increase of \( g \). In fact, from \( U = Kg^{a}r^{b} \) and \( a = pb \) we get

\[
dU = dg \left( pbKg^{a-1}r^{b} \right) + dr \left( bKg^{a}r^{b-1} \right)
\]

so, if \( U \) does not change, so that \( dU = 0 \), then

\[
dg \left( pbKg^{a-1}r^{b} \right) = -dr \left( bKg^{a}r^{b-1} \right) \quad \text{or} \quad \frac{dg}{dr} = -p \frac{dr}{g}.
\]

The only a priori information we need to assess the fulfillment of the condition, is the value of \( p \), the ratio of the elasticities. We argue that this parameter may actually be provided by the political system. The ratio may be considered a proxy of the relative weight that the community assigns to an increase in the relative value of \( g \) and \( r \). If for instance a – say – 10% increase in \( g \) is valued more than the same increase in \( r \), \( p > 1 \), and vice versa.\(^{10}\) To keep into account the specific form of the indices, one may resort to the so-called decision maker approach: the index \( g \) (more arbitrary than \( r \)) may be scaled, and the amplitude of the scaling requested to produce a change of choice may be assessed.

Equations (2a) and (2b) allow for binary comparisons of (non-dominated) electoral systems, and hence for finding out the Condorcet winner. The winner is the best system.\(^{11}\)

Alternatively, we may trace indifference curves and pick the system that lies on the higher curve. The expression for the generic indifference curve \( r = r(U, g) \) is:

\[
r = \left( \frac{U}{K} \right)^{\frac{1}{a}} \left( \frac{g}{g^*} \right)^{\frac{b}{a}}
\]

(3)

or

\[
r = \frac{W}{g^*}
\]

(4)

\(^{10}\) Remember that this ratio is constant along an indifference curve.

\(^{11}\) A Condorcet cycle may result only by chance, and may be ruled out simply by adding a further figure while rounding the results.
where $W = \left( \frac{U^*}{K} \right)^{\frac{1}{g}}$.

This provides an alternative way to compare pairs of systems. For a given value of $g$, the value of $r$ increases with that of $W$, and the value of $W$ increases with that of $U^*$. Consequently, it is sufficient to solve equation (4) for $W$, given $r$, $g$, and $p$, for each system considered. The system with the highest value of $W$ is the best one. The solution may appear graphically, as in Figure 3, where indifference curves for a hypothetical value of $p$ have been plotted across the (hypothetical) data of Figure 1. In absence of $?$, and ruling out the dominated system 4, the winner is system 2.

Just to give an example, we will illustrate the results of some simulations, run at the Laboratory of Experimental and Simulative Economics of the Università del Piemonte Orientale. The Parliament is supposed to have 100 members. The voters are obtained from a representative, nation-wide survey of the complete preferences for (then) existing parties of Italian citizens in 1997.12 We always assumed that the majority was a minimal winning coalition of adjacent parties on the left-right axis. The method employed is that of

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12 This looks somehow at odds with our previous claim that survey data are of no use, but it isn’t. The Italian electoral system was perfect proportionality (PP), with constituencies of approximately 20 seats. In 1994, it was changed to a mixed-member system. Hence the voters had to express the order of their real preferences also for the PP, in a setting characterized by a long history of a large number of parties. From this order it is possible to infer the behavior for other systems, through suitable aggregations; what is not possible is to infer from a real (say) plurality setting the preferences for PP or other proportional systems. Data were collected by ISPO (Institute for the Study of the Public Opinion), Milano.
equation (4) above; hence the crucial figure is the value of \( W \).

Results are summarized in the table below, where the systems are the same as in Example 1. Remember that if \( p > 1 \), governability is more appreciated than representativeness, and vice versa if \( p > 1 \). With \( p = 1 \), the winner is prized proportionality. Remarkably, the same holds for \( p = 2 \): assigning governability twice the value of representativeness is not enough to make non-proportional systems preferable, at least for Italian voters. Not unexpectedly, if \( p = 0.5 \) the best system is proportionality. Mixed-members systems perform quite well, thus confirming some recent suggestions (see Shugart, 2001, or Shugart and Wattenberg, 2000).

### 4. Concluding remarks

We argued that the simulation approach to the evaluation of electoral systems is very powerful. To add evidence, we suggest a (very partial) list of problem that could profitably be tackled this way. What is the difference in results between Borda and Condorcet? When do pure proportionality and single transferable vote provide analogous results? What is the actual effect of district magnitude on proportionality? How do indices of proportionality perform? Are Condorcet cycles really a problem?

It is not difficult to add others, so we will not pursue this point further. Instead, we argue that experimental results may be improved if the simulation programs are further elaborated. We suggest that main methodological improvements should regard the possibility of including and managing...
survey data, the addition of further indices, mostly but not only with reference to power issues, and obviously the addition of further electoral systems. However, to our opinion the main methodological challenge is the addition of new evaluation dimensions, and consequently indicators. Obviously, this requires that they may be quantified, and consensus on what we desire about.

To conclude, simulation is very useful to analyze the performance of electoral systems, and it provides very interesting results. The most interesting ones lay probably ahead.

Appendix A  Power and governability

Throughout the paper (and the simulation program) we assumed that the governability is inversely related to the number of parties that support the government. Despite its common acceptance, the theoretical foundation of this assumption is not sound; more intriguingly, the empirical confirmation is quite poor too. It is not a surprise that some major authors, like Lijphart (1999) or Farrell (2001), reject it.

In our opinion, a better definition of governability should rest on the notion of power. We plan to substitute $G$ with a power-based index in next versions of the simulation program. However, in order to do that it is necessary to tackle quite a lot of problems. We think that it is worthy to discuss some of them here, albeit our analysis is still very preliminary.

The elusive notion of power has a lot to do with the choice of the electoral system; and both with governability and with representativeness. If we stick to the microcosm notion of representativeness, we should want a distribution of power similar to that of preferences; while the governability is normally supposed to be enhanced if the power is highly concentrated. To find out the ‘right’ distribution of power is a formidable task, and we will not deal with it. More modestly, we argue that in order to tackle that problem it is necessary to be able to compare the distribution of power with that of preferences; and again simulation is highly useful, for the same reasons that we discussed above – the non-availability of reliable real world data.

More precisely we face two problems, strongly related one another: The first is to determine the power of a party on the basis of the distribution of voters and on the basis of the distribution of seats in the parliament, the second is to measure the difference of the distribution of power in the above-mentioned situation or, in other words, to measure the distance of the two distributions.

Game theory is a natural habitat for the problem of evaluating the power of the parties in a voting situation. Since the pivotal paper of Shapley (1953)
different indices were introduced, with the aim of assigning to each agent a number that represents his/her relevance in a multiagent situation. Before moving to our results, it may be useful to recall some basic notions. A cooperative game with transferable utility (TU-game) is a pair $G=(N,v)$, where $N$ is the set of players (the agents) and $v$ is the characteristic function that assigns to each subset of players $S \subseteq N$, called coalition, a real number that can be considered as its worth independently from the behavior of the other players. A game is said to be simple if $v(S) \in \{0,1\}$, i.e. the worth of a coalition may be only 0 or 1; a game is said to be monotonic if $S \subseteq T$ implies $v(S) \leq v(T)$, i.e. if a coalition is enlarged then its worth cannot decrease. In particular we are interested in the weighted majority games. Weighted majority games are simple monotonic games that are widely used in voting situations. Suppose that each player $i \in N$ is associated with a non-negative real number, the weight $w_i$, and suppose that if some players join to form a coalition $S$ the weight of the coalition is the sum of the weights of the players, i.e. the weights are additive; if the weight of a coalition is strictly larger than a given positive real number $q$, the so-called quota, the coalition is said to be winning, and it is said to be losing otherwise. Formally we define the characteristic function $w$ of a weighted majority game as:

$$w(S)=\begin{cases} 1 \text{ if } \sum_{i \in S} w_i > q, \forall S \subseteq N \\ 0 \text{ if } \sum_{i \in S} w_i \leq q \end{cases}$$

Usually such a situation is summarized by the $(n+1)$-upla $(q,w_1,\ldots,w_n)$.

As a consequence we can say that if $v(S)=1$ then $S$ is a winning coalition and if $v(S)=0$ then $S$ is a losing coalition. A winning coalition is called minimal if all its subcoalitions are losing.

The weighted majority games associated to the distributions of voters and of seats, according to a given electoral system, allow us evaluating the importance of each party with respect to a suitable power index. Game theory dealt with this problem from the beginning of its history. Many different power indices were proposed, each of them emphasizing different properties of the underlying situation. In this paper we consider the Shapley-Shubik index, the normalized Banzhaf-Coleman index, the Deegan-Packel index and the Holler (or Public goods) index.

The Shapley-Shubik index (Shapley and Shubik, 1954), $\varphi$, is the natural extension of the Shapley value (Shapley, 1953) to simple games. Consider the set $\Pi$ of all the orderings (permutations) of the players and for each ordering $\pi \in \Pi$ let $P(i,\pi)$ be the set of players that precede player $i$ in the ordering $\pi$; the Shapley value is the average marginal contribution of each player w.r.t.
the possible orderings:

$$
\varphi_i = \frac{1}{|N|} \sum_{\pi \in \Pi^{N-i}} [v(P(i, \pi) \cup \{i\}) - v(P(i, \pi))], \forall i \in N
$$

where $|N|$ denotes the cardinality of the set of players $N$.

The normalized Banzhaf-Coleman index (Banzhaf, 1965, and Coleman, 1971), $\beta$, is similar to the Shapley-Shubik index, but it considers all the marginal contributions of a player to all possible coalitions, without considering the order of the players. Let us introduce:

$$
\beta_i = \frac{1}{2^{N-1}} \sum_{S \subseteq \Pi^{N-i}} [v(S) - v(S \setminus \{i\})], \forall i \in N
$$

where again $|N|$ denotes the cardinality of the set of players $N$.

By normalization we get:

$$
\beta_i = \frac{\beta_i^*}{\sum_{j \in N} \beta_j^*}, \forall i \in N
$$

The Deegan-Packel index (Deegan and Packel, 1978), $\delta$, considers only the minimal winning coalitions; the power is firstly equally divided among minimal winning coalitions and then the power of each is equally divided among its members. Let $W = \{S_1, \ldots, S_m\}$ be the set of minimal winning coalitions; formally:

$$
\delta_i = \sum_{S_k \in W, S_k \subseteq \Pi^{N-i}} \frac{1}{|W||S_k|}, \forall i \in N
$$

where $|W|$ and $|S_k|$ denote the cardinalities of the set of minimal winning coalitions $W$ and of the coalition $S_k$, respectively.

The Holler index (Holler, 1982, and Holler and Packel, 1983), $H$, or Public Goods index is defined as follows. First, we consider the number $c_i$, $i \in N$, of minimal winning coalitions which player $i$ belongs to, then, by normalization, we get:

$$
H_i = \frac{c_i}{\sum_{j \in N} c_j}, \forall i \in N
$$
Using these indices, we can measure the distance of an electoral system $h$ from the distribution of preferences, computing the power indices according to the distribution of votes in pure proportionality (we assume that it corresponds to that of preferences, as we already did in Example 1), $v$, and to the assignment of seats in that system, $s^h$.

We refer to the three most widely used distances or norms:

- **Norm 1** (sum of the absolute values of the differences among the percentages of votes and of seats):
  
  $$d_v^h = \sum_{i \in N} |v_i - s_i^h|$$

- **Norm 2** or Euclidean distance (square root of the sum of the squares of the differences among the percentages of votes and of seats):
  
  $$d_v^h = \sqrt{\sum_{i \in N} (v_i - s_i^h)^2}$$

- **Norm $\infty$** (maximum of the absolute values of the differences among the percentages of votes and of seats):
  
  $$d_v^h = \max_{i \in N} |v_i - s_i^h|$$

**Example 2** Suppose that there are four parties $P_A$, $P_B$, $P_C$ and $P_D$; the preferences of the voters are respectively 40, 25, 20 and 15 per cent and the majority quota is 50 per cent; suppose also that the parliament consists of 4 seats and that two voting systems generate the two distributions $(2,1,1,0)$ and $(1,1,1,1)$. We start by computing the distances of the two distributions of seats from the distribution of voters:

<table>
<thead>
<tr>
<th>Distribution</th>
<th>$(2,1,1,0)$</th>
<th>$(1,1,1,1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1$</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>$d_2$</td>
<td>0.01 $\sqrt{350}$</td>
<td>0.01 $\sqrt{350}$</td>
</tr>
<tr>
<td>$d_\infty$</td>
<td>0.15</td>
<td>0.15</td>
</tr>
</tbody>
</table>

The two seat distributions seem to be equivalent, so we consider the majority games $w(v)$ on voters, $w(s^1)$ on the first parliament and $w(s^2)$ on the second parliament.
whose corresponding indices are:

Table 6

<table>
<thead>
<tr>
<th>game</th>
<th>φ</th>
<th>β</th>
<th>δ</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>w(v)</td>
<td>1111</td>
<td>1111</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>w(s)</td>
<td>1111</td>
<td>1111</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>w(z)</td>
<td>1111</td>
<td>1111</td>
<td>24</td>
<td>24</td>
</tr>
</tbody>
</table>

Finally, for each index, we compute the distances between the power w.r.t. the voters and to each parliament:

Table 7

<table>
<thead>
<tr>
<th>(2,1,1,0)</th>
<th>(1,1,1,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>dφ</td>
<td>0.333</td>
</tr>
<tr>
<td>dβ</td>
<td>0.417</td>
</tr>
<tr>
<td>dδ</td>
<td>0.444</td>
</tr>
<tr>
<td>dH</td>
<td>0.167</td>
</tr>
</tbody>
</table>

The distances on the power indices allow us to distinguish the two systems.

Another measure may be obtained if we refer to the percentages of distribution of voters, \( v \), to the assignment of seats according to an electoral system \( h \), \( s^h \), and to the power of the parties related to the votes and to the seats, \( \psi \) and \( \psi^h \) respectively (see Gambarelli and Biella, 1992). The resulting distance \( \Delta \) is:
\[ \Delta = \max_{n \in N} \left\| \psi_n - \mathbf{s}_n \right\| - \left\| \psi - \mathbf{s} \right\| \]

In the following example we apply this measure of the distance to a hypothetical situation obtained from a simulation round. It shows that the distance tends to be lower in proportional systems – but remember that this is an exemplum fictum.

**Example 3** Referring to the data of Example 1 the distances are:

<table>
<thead>
<tr>
<th>Voting system</th>
<th>( \psi = \varphi )</th>
<th>( \psi = \beta )</th>
<th>( \psi = \delta )</th>
<th>( \psi = H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P - 4 )</td>
<td>0.013</td>
<td>0.013</td>
<td>0.057</td>
<td>0.063</td>
</tr>
<tr>
<td>( P(20) )</td>
<td>0.286</td>
<td>0.161</td>
<td>0.067</td>
<td>0.084</td>
</tr>
<tr>
<td>( M )</td>
<td>0.136</td>
<td>0.099</td>
<td>0.088</td>
<td>0.094</td>
</tr>
<tr>
<td>( 2R )</td>
<td>0.136</td>
<td>0.099</td>
<td>0.088</td>
<td>0.094</td>
</tr>
<tr>
<td>( C )</td>
<td>0.097</td>
<td>0.116</td>
<td>0.116</td>
<td>0.130</td>
</tr>
<tr>
<td>( B )</td>
<td>0.146</td>
<td>0.146</td>
<td>0.059</td>
<td>0.072</td>
</tr>
<tr>
<td>( A )</td>
<td>0.176</td>
<td>0.176</td>
<td>0.086</td>
<td>0.106</td>
</tr>
<tr>
<td>( I - 25 )</td>
<td>0.113</td>
<td>0.109</td>
<td>0.157</td>
<td>0.161</td>
</tr>
<tr>
<td>( I - 75 )</td>
<td>0.029</td>
<td>0.029</td>
<td>0.043</td>
<td>0.087</td>
</tr>
</tbody>
</table>

Numbers in bold indicate the best voting system according to each index.

The main conclusions of this appendix are the following. The indication of the example – were it for real – would be precious. Yet the starting point is, by necessity, the data on votes. If votes are those actually cast in a, say, plurality election, they are useless to compare the distribution of power with that of preferences. Arguably, the distribution of votes may be assumed as a proxy to that of preferences only in proportional systems with large districts (and, we must add, with low running costs). The same conclusions of Section 3 apply. Real data cannot provide useful information; the simulation does. To accumulate experimental (i.e. simulative) evidence would probably provide relevant suggestions for real world analysis and policing.
Appendix B  The indices employed in the simulation programs

B1. Index of representativeness, $r$

For the reasons exposed at length above, a suitable index of representativeness cannot be based on the difference between the share of votes and that of seats, as all the indices of proportionality commonly employed, like Gallagher’s, do. Instead, our index is based on the difference between votes cast in a nation-wide proportional district and seats assigned by a given electoral system. The formula is:

$$r_h = 1 - \frac{\sum_{i \in N} |S^i_h - S^{op}_i|}{\sum_{i \in N} |S^i_u - S^{op}_i|}$$

where $N$ is the set of parties, $S^i_h$ is the number of seats of party $i$ with system $h$, $S^{op}_i$ is the number of seats of party $i$ with the perfect proportional system and $S^i_u$ is the total number of seats for the relative majority party under system $h$ and it is 0 otherwise.

The index reads as follows. For the sum at the numerator, we assume that the representativeness $R$ is maximal under perfect proportionality rule ($PP$). Hence the loss of representativeness incurred by party $i$ is the (absolute) difference between the seats it would get under $PP$ and those actually obtained. Summing this loss across all the parties we obtain the total loss of $R$. The sum at the denominator is introduced to normalize this value. It is the maximum possible loss of $R$. This maximum is obtained when ‘winner takes all’ in a very strict sense, that is when the relative majority party, according to the selected system, takes all the seats instead of just its quota. The ratio of the sums is a loss of representativeness index, normalized in the range 0–1; subtracting it from 1 we transform it into a representativeness index.

Example 4  Suppose three parties, $L$, $C$ and $R$, in a parliament of 100 seats. Under $PP$ they obtain 49, 31 and 20 seats respectively, under majority ($M$) 90, 10 and 0, and under some other system ($S$) 60, 25 and 15. So

$$r_M = 1 - \frac{41 + 21 + 20}{51 + 31 + 20} = 0.196 \quad \text{and} \quad r_S = 1 - \frac{11 + 6 + 5}{51 + 31 + 20} = 0.784$$

obviously  \( r_{sp} = 1 - \frac{0}{51 + 31 + 20} = 1 \)

**B2. Index of governability, \( g \)**

According to the mainstream doctrine (but not everyone agrees, as we saw), governability is inversely related to the number of parties that take part in the governing majority. Our index is based on this assumption. It depends on the number of parties of the governing coalition that may destroy the majority if they withdraw, \( m \), and on the share of seats of the majority, \( f \). \( m \) is more important, so we add (lexicographically) the \( f \)-component to the \( m \)-component. Hence the index is made by the sum of two terms, the first related to \( m \), \( g_m \), and the second related to \( f \), \( g_f \). Thus, \( g = g_m + g_f \). The range of the second term is the difference between successive values of the first: the term in \( m \) defines a lower and an upper bound, and the term in \( f \) specifies the value of the index between them.

The range defined for \( g \) is simply \( 1/m \) (upper bound) and \( 1/m + 1 \). For instance, if the government is supported by just one party, \( g \) is in between 0.5 and 1 if it supported by two parties, then \( g \) is in between 0.333 and 0.5, and so on. The number of seats of the majority coalition specifies the value of \( g \) in the given range. The amount \( g_f \) to be added to the lower bound depends on the lead of the majority coalition, according to the following proportion:

\[
\frac{g_f}{m - \frac{1}{m+1}} = \frac{f - \frac{T}{T}}{T - \frac{T}{T}}
\]

from which:

\[
g_f = \frac{1}{m(m+1)} \frac{f - \frac{T}{T}}{T}
\]

where \( T \) is the total number of seats in the Parliament.

In sum, the formula for \( g \) is:

\[
g = g_m + g_f = \frac{1}{m+1} \frac{1}{m(m+1)} \frac{f - \frac{T}{T}}{T}
\]
For instance, if there are 100 seats and the governing majority is made up of one party with 59 members, we have \( g_f = 9/50 \cdot 1/2 = 0.09 \). This value must be added to 0.5, to give \( g = 0.59 \).

The maximum value of \( g \) is 1, when a party has all the seats; the lowest tends to zero as the number of parties increases, thus justifying the claim that \( g \) is in the range of \( 0 \rightarrow 1 \).

**Appendix C  The electoral systems considered**

The symbols are those used in the paper. Most systems admit several versions; the definition refers to the one employed in this paper. We assumed a unique 100-seat constituency for all the proportional systems, but for the mixed-member system where seats were 25 or 75 plus 75 or 25 one-seat constituencies, and 100 one-seat constituencies for all the others.

**Pure Proportionality, \( P \)** The seats are assigned to the parties according to their (rounded) shares of votes.

**Threshold Proportionality, \( P-n \)** The seats are assigned to the parties according to their (rounded) shares of votes, with the exclusion of the parties that obtained less than \( n \) percent of the votes.

**Prized Proportionality, \( P(n) \)** The largest party obtains a bonus of \( n \) seats.

**Relative Majority (also called Plurality or First-Past-The-Post), \( M \)** Each constituency elects the candidate that obtains the largest share of votes.

**Two-round Run-off (also called Run-off Majority), \( 2R \)** If no candidate obtains an absolute majority of votes in her/his constituency, the two that obtained the largest shares of votes compete in a second round.

**Condorcet Method, \( C \)** In each constituency, the winner is the candidate that obtains more votes than each of the others in pairwise voting. Note that the winner may not exist; this case never occurred in our simulations. Note also that the computing of the Condorcet winner requires the knowledge (here the simulation) of the full ordering of preferences of all the voters. The same holds for the next system.

**Borda Count, \( B \)** Each voter assigns a score to each candidate in her/his constituency, ranging from 0 (to the best preferred candidate) to \( n \) (to the...
last preferred). The candidate who achieves a minimum of scores is elected.

**Approval Voting.** A  Each voter votes for her/his $k$ preferred candidate in her/his constituency, out of a list of $n$ candidates, with $k \leq n$. The candidate who gets more votes is elected. (In this paper the value $k$ is assigned randomly).

**Mixed-member System, $I - n$** A given share $n$ of the Parliament is elected through pure proportionality, the remaining share through relative majority.

**Acknowledgments**

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