A note on “Measurement of disproportionality in proportional representation systems”

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Abstract

In this paper, we add three further indices to the survey on disproportionality by Karpov. Two of them are rooted in the issue of power, so we discuss the relevance of it in evaluating the disproportionality of a Parliament w.r.t. the voting body.

Keywords: disproportionality indices, power, power indices, simple games
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1. Introduction

In [12] nineteen disproportionality indices are presented and axiomatically analyzed; then, they are applied to four electoral sessions in Russia for analyzing the features of these indices when the number of parties varies. Electoral systems and their features, mainly proportionality and the related problem of apportionment, are widely studied in literature. We just mention the contributions on electoral systems by Brams and Fishburn [2], Hołubiec and Mercik [11], Nurmi [15] and Lijphart [13], while more specific on proportionality are the papers by Gallagher [6] and Monroe [14] and the analysis of apportionment methods by Gambarelli and Palestini [8]. In our opinion, the paper by Karpov is a very good commented survey on disproportionality indices, so we think that it could be of interest to analyze also the indices proposed by Ortona [4], Fragnelli [3] and Gambarelli and Biella [7]. In particular, the last two indices account the issue of power

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for measuring the disproportionality. In our mind, the power, i.e. the influence of each party on the decision of passing a law, should play a more relevant role in evaluating the characteristics of a Parliament, including the proportionality (or representativeness). As we point out in Section 3 with suitably designed academic examples, it is possible that an apparently unfair distribution of seats w.r.t. votes may provide the parties the same power of their voters, so we can conclude that the voting body is well represented by the Parliament. This is due mainly to two reasons: the first is that the so-called proportional systems introduce some modifications in order to enhance other good features, *in primis* the governability, excluding the smallest parties, via a threshold, and/or strengthening the largest parties, via a majority prize; the second is that even with a perfect proportional system it is necessary to assign an integer number of seats and the rounding methods may change the power of the parties. The paper is organized as follows: in the next section we recall the three added indices; in Section 3 we present two academic examples to show the relevance of the issue of power; Section 4 contains the value assigned by the three indices to the Russian electoral sessions used by Karpov and a discussion on the comparison with the indices in [12]; Section 5 concludes.

2. The added indices of disproportionality

For completing the analysis of the disproportionality indices and of their properties, we add three indices: the Ortona index, the Fragnelli index and the Gambarelli-Biella index. The first two ones are defined as representativeness indices, but both of them are obtained as one minus a disproportionality index, so we present them as disproportionality indices, for consistency with the paper of Karpov.

Ortona index. It was proposed in [4] and it is based on the difference between seats assigned by a given electoral system and seats assigned by a perfect proportional system, *PP*, i.e. supposing a unique nation-wide proportional district and assigning the rest to the largest decimals. The idea is to avoid combining votes and seats as most of the indices of disproportionality commonly do. The formula is:

$$r = \frac{\sum_{i \in N} |S_i - S_i^{PP}|}{\sum_{i \in N} |S_i^u - S_i^{PP}|}$$

where N is the set of parties, S_i is the number of seats of party i with the system under consideration, S_i^{PP} is the number of seats of party i with the perfect proportional system and S_i^u is the total number of seats for the relative majority party and 0 otherwise.

The index reads as follows. For the sum at the numerator, we assume that the disproportionality is minimal under perfect proportionality rule. Hence, the loss of disproportionality incurred by party i is the (absolute) difference between the seats actually obtained and those it would get under PP. Summing this absolute difference across all the parties we obtain the total disproportionality. The sum at the denominator is introduced to normalize this value. It is the maximum possible disproportionality obtained when the relative majority party, according to the selected system, takes all the seats instead of just its quota. Let T be the total number of seats and, without loss of generality, party 1 be the relative majority party; then $\sum_{i\in N}|S_i^u - S_i^{PP}| = |S_1^u - S_1^{PP}| + \sum_{i\in N\setminus\{1\}}|S_i^u - S_i^{PP}| = T - S_1^{PP} + \sum_{i\in N\setminus\{1\}}|-S_i^{PP}| = 2(T - S_1^{PP})$.

The ratio of the sums is a disproportionality index, normalized in the range $[0,1]$.

Note that this index cannot be employed starting from real-world votes in a non-proportional system, due to strategic voting. However, some ingenuity could allow for using it starting from survey data.

Fragnelli index. The second index we want to add was introduced in [3]. The idea is that the issue of power plays a relevant role in evaluating the representativeness of a Parliament.

In order to deal with the concept of power, starting from the vote share $(v_1, \ldots, v_n)$ and the seat share $(s_1, \ldots, s_n)$, we define two simple games (N, w) and (N, u), where $N = \{1, \ldots, n\}$ is the set of parties and w and u are the two characteristic functions $w, u : 2^N \to [0,1]$; w is valued 1 for the winning coalitions of parties, i.e. coalitions with a total vote share greater than 0.5, sufficient to pass a law, and 0 for the loosing coalitions; similarly for the function u referred to the seat share.

A normalized power index is a map φ that assigns to each simple game (N, v) a non negative real vector $\varphi(v)$ with $\sum\limits_{i\in N}\varphi_i(v)= 1$. Many power indices have been defined, for measuring the power of a party in a Parliament.

This disproportionality index measures the distance of the distribution of power on the votes and on the seats, i.e. $\sum_{i\in N}|\varphi_i(w) - \varphi_i(u)|$. It is normalized, simply dividing $\sum_{i\in N}|\varphi_i(w) - \varphi_i(u)|$ by 2, as in the worst case the two distributions of power may assign complementary values [1]. So, we have:

$$r^\Omega = \frac{\sum_{i\in N}|\varphi_i(w) - \varphi_i(u)|}{2}$$

Gambarelli-Biella index. The last index, proposed in [7], is a combination of the traditional approach, which considers vote and seat shares, with the idea of measuring the distance of the distributions of power related to the votes and to the seats. It is given by:

$$\Delta = \max_{i\in N}\{|v_i - s_i|, |\varphi_i(w) - \varphi_i(u)|\}$$

[1] Two vectors are complementary when each non-zero component in a vector corresponds to a zero component in the other vector.

Remark 1. *Note that r^{Ω} is based on norm 1, while Δ is based on norm ∞, so it is possible to define other indices based on other norms.*

Referring to the principles given by Karpov, *anonymity*, *transfers*, *independence from split* and *scale invariance* (see Section 8 in [12]), we can say that Ortona index satisfies all these four properties while Fragnelli index and Gambarelli-Biella index satisfy the first and the last one, but not the other two ones. Karpov underlines the fact that violation of property 3 means that the index depends on the number of parties. In our analysis, the violation of this property is more related to a variation of the distribution of the power inside a Parliament; analogously, the transfer of seats from an overrepresented party to an underrepresented one, on the one hand, may have a positive effect on the distribution of seats w.r.t. to votes, but on the other hand may dramatically change the power distributions.

3. Academic Examples

The situations we are going to analyze, even if they are not very realistic, mainly for the small number of seats and the distribution of votes, are useful to show how the indices based on power focus on a totally different point, compared with the traditional ones. Parliaments which are very good, evaluated by the classical indices, can be very bad from the point of view of representing the power and vice versa. More precisely, we refer to the following 14 disproportionality indices (grouped according the ranges of values they may assume):

indices in $[0,1]$: Maximum Deviation Index (MD), Loosmore-Hanby Index (I_{LH}), Lijphart Index (I_L), Gallagher Index (Lsq), Gini Index (G);

indices with non-negative range of values (different from $[0,1]$**)**: Rae Index (I_{Rae}), Grofman Index (I_{Gr}), Monroe Index (I_M), Gatev Index (I_{Ga}), Ryabtsev Index (I_R), Szalai Index (I_S), Sainte-Lague Index (SL);

indices with values larger than 1: Aleskerov-Platonov Index (R), D'Hondt Index (H).

We refer to [12] for the descriptions and the formal definitions of the indices. We omit the Atkinson Index and Generalized Entropy Index, as they require fixing some parameters, and some variations of some previous indices.

The different ranges of values do not allow for a cross-analysis, but the following situations are designed in such a way that all the abovementioned indices and the index by Ortona produce a result in contrast with the two power-based indices. For computing the last two indices we have to decide which power index to use. There exists a wide literature, including dozens of indices so we chose two of the most popular ones, namely the Shapley-Shubik index [16] and the Public Good Index [9]. Our choice depends on their different behavior; in fact, the former has the local monotonicity property, so that it assigns larger power to parties with greater number of seats, while the latter do not respect monotonicity and the difference of power are often very small w.r.t other non monotonic indices (see [10]).

The first one, denoted by ϕ, is based on the concept of marginal contribution and it is given by the following formula

$$\phi_i(w) = \sum_{S \in W, S \ni i} \frac{(|S| - 1)!(n - |S|)!}{n!}[w(S) - w(S \setminus \{i\})] \quad \forall i \in N$$

where W is the set of winning coalitions.

The second one, denoted by PGI, is based on the minimal winning coalitions, i.e. coalitions which become losing whatever a party leaves. The formula is given by

$$PGI_i(w) = \frac{h_i}{\sum_{k \in N} h_k} \quad \forall i \in N$$

where h_i is the number of minimal winning coalitions including party $i \in N$. Again, we refer to the original papers for the descriptions and the formal definitions of the indices.

3.1. Academic Example 1

We start revisiting the situation in Example 1 in [3]. Three parties A, B, C receive a percentage of votes of $49.5\%, 48.5\%, 2.0\%$, respectively and 6 seats have to be assigned. We consider three different seats assignments:

	A	B	C
s_{PP}	3	3	0
s	3	2	1
s_{PO}	2	2	2

Table 1 - Seat distribution for academic example 1

s_{PP} assigns the seats according the perfect proportional system; s guarantees one seat to party C, with the consequence that party A receives one seat more than party B; finally, s_{PO} (PO after Power Oriented) guarantees a distribution of power on the seat sharing equal to the distribution of power on the vote sharing, even if it does not seem as reasonable as the other two.

Computing the abovementioned disproportionality indices we obtain the following table:

	MD	I_{LH}	I_L	Lsq	G	I_{Rae}	I_{Gr}	I_M	I_{Ga}	I_R	I_S	SL	R	H
s_{PP}	0.020	0.020	0.010	0.018	0.020	0.013	0.019	0.021	0.026	0.018	0.577	0.021	1.021	1.031
s	0.152	0.152	0.078	0.149	0.228	0.101	0.146	0.173	0.226	0.162	0.587	1.123	1.010	8.333
s_{PO}	0.313	0.313	0.157	0.271	0.313	0.209	0.301	0.315	0.425	0.315	0.598	5.009	16.667	16.667

Table 2 - Disproportionality indices in [12] for academic example 1

The values confirm that the assignment s_{PO} is the worst under all the indices, having always the largest value in each column.

Next, we compute the Shapley-Shubik index and the Public Good index referring to the three seat distributions and to the vote sharing, v:

	A	B	C
$\phi(v)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\phi(s_{PP})$	$\frac{1}{2}$	$\frac{1}{2}$	0
$\phi(s)$	$\frac{4}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$\phi(s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$PGI(v)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$PGI(s_{PP})$	$\frac{1}{2}$	$\frac{1}{2}$	0
$PGI(s)$	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$PGI(s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Table 3 - Power indices for academic example 1

Computing the indices by Ortona, Fragnelli and Gambarelli-Biella we obtain the following table:

	r	r^{Ω}		Δ	
		ϕ	PGI	ϕ	PGI
s_{PP}	0	0.333	0.333	0.333	0.333
s	0.333	0.333	0.167	0.333	0.167
s_{PO}	0.667	0	0	0.313	0.313

Table 4 - Added disproportionality indices for academic example 1

The index by Ortona that does not account the power assigns the worst value to assignment s_{PO}, that on the other hand receives the best score under the index by Fragnelli that is strongly power-oriented; the index by Gambarelli-Biella is influenced by the seat assignment.

3.2. Academic Example 2

In this second situation, we consider again three parties A, B, C that receive a percentage of votes of $49.9\%, 25.2\%, 24.9\%$; we analyze two different Parliaments with 9 and

10 seats, respectively; finally we compute the assignment of the seats under PP (s_{PP}) and looking at the power distribution (s_{PO}):

	A	B	C
$9 - s_{PP}$	5	2	2
$9 - s_{PO}$	4	3	2
$10 - s_{PP}$	5	3	2
$10 - s_{PO}$	4	3	3

Table 5 - Seat distribution for academic example 2

Note that assignment s_{PP} gives to party A the absolute majority in the Parliament with 9 seats and the veto power with 10 seats, differently from the distribution of votes and assignment s_{PO}. Computing the disproportionality indices in [12] we obtain the following table:

	MD	I_{LH}	I_L	Lsq	G	I_{Rae}	I_{Gr}	I_M	I_{Ga}	I_R	I_S	SL	R	H
$9 - s_{PP}$	0.057	0.057	0.043	0.049	0.057	0.038	0.042	0.059	0.078	0.056	0.579	0.013	1.113	1.113
$9 - s_{PO}$	0.081	0.081	0.068	0.072	0.082	0.054	0.061	0.087	0.119	0.084	0.584	0.035	1.323	1.323
$10 - s_{PP}$	0.049	0.049	0.025	0.049	0.073	0.033	0.037	0.059	0.079	0.056	0.580	0.019	1.096	1.190
$10 - s_{PO}$	0.099	0.099	0.074	0.086	0.100	0.066	0.074	0.103	0.143	0.102	0.583	0.039	1.198	1.205

Table 6 - Disproportionality indices in [12] for academic example 2

Again, the assignment s_{PO} is the worst under all the indices, with both 9 and 10 seats. The Shapley-Shubik index and the Public Good index are:

	A	B	C
$\phi(v)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\phi(9 - s_{PP})$	1	0	0
$\phi(9 - s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\phi(10 - s_{PP})$	$\frac{4}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$\phi(10 - s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$PGI(v)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$PGI(9 - s_{PP})$	1	0	0
$PGI(9 - s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$PGI(10 - s_{PP})$	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$PGI(10 - s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Table 7 - Power indices for academic example 2

Computing the indices by Ortona, Fragnelli and Gambarelli-Biella we obtain the following table:

	r	r^{Ω}		Δ	
		ϕ	PGI	ϕ	PGI
$9 - s_{PP}$	0	0.667	0.667	0.667	0.667
$9 - s_{PO}$	0.250	0	0	0.081	0.081
$10 - s_{PP}$	0	0.333	0.167	0.333	0.167
$10 - s_{PO}$	0.200	0	0	0.099	0.099

Table 8 - Added disproportionality indices for academic example 2

Again, the index by Ortona assigns the worst value to assignments s_{PO}, that receive the best score under the two power-oriented indices.

4. Application to the Russian Parliament

In [12] the proposed disproportionality indices are computed for the elections to the State Duma (Russian Parliament) referring to four electoral sessions: 1995, 1999, 2003 and 2007. The results are shown in Tables 10.1 and 10.2 of the cited paper. In the following table we show the results for the three added indices, completing the analysis. Due to the high number of parties, we omit the values of the Shapley-Shubik index and of the Public Good index. The computation was carried out with the classical generating functions algorithm for the Shapley-Shubik index [1] and for the Public Good index with a new algorithm based on the generating functions, designed by Michela Chessa, available upon request.

	r	r^{Ω}		Δ	
		ϕ	PGI	ϕ	PGI
1995	0.6129	0.4298	0.9063	0.2178	0.3098
1999	0.1843	0.0624	0.3818	0.0503	0.1741
2003	0.4170	0.4213	0.9548	0.4213	0.9548
2007	0.2038	0	0	0.0498	0.0498

Table 9 - Added disproportionality indices for Duma

Analyzing the indices, Karpov observed that almost all of them create the same ordering, that the Parliament of 1999 is the most proportional and the Parliament of 1995 is the least one. This is still true for the results obtained evaluating the Ortona index, and we can notice that, also computing the other two indices that take into account the role of the power, the Parliament of 1999 has a very low disproportionality value, but the best score is given to the Parliament of 2007. Fragnelli index accounts it for perfect proportionality; this is due to the particular situation that *United Russia* got the absolute majority both on the vote share (64.30%) and on the seat share (70.00%), so it got the whole power and the differences on vote and seat shares have no relevance at all.

This fact underlines that, even if from the academic examples we have shown that to evaluate disproportionality taking into account the power can give very different results than evaluating it with classical indices, it is possible that a Parliament has a good evaluation from both points of view, enforcing the approaches and showing that they are not necessarily conflicting.

5. Concluding Remarks

The different values of the indices for the different Parliaments suggest that they are suitable for measuring the disproportionality more of a Parliament than of the adopted electoral system.

The issue of power allows new perspective for analyzing the disproportionality. The situation in academic example 1 is designed in order to have the maximal impact on the reader, using a very limited number of seats. We want to make clear that with a higher number of seats it is possible to reduce the differences in the values of the indices for the three assignment systems. Moreover, the equal distribution of power on the votes and on the seats could be obtained with a very reasonable distribution of seats. For instance, using the same percentages of votes with 15 seats, the assignment rules s_{PP} and s correspond to $(8,7,0)$ and $(7,7,1)$, respectively, and the first is easily rejected because Party A would have the absolute majority. If we want to guarantee the same distribution of power on the votes and on the seats it is sufficient to assign to each party a number of seats greater than or equal to 1 and less than or equal to 7, i.e. s_{PO} can be selected equal to s. This remark may be exploited for designing a rule for assigning the rests in a proportional system (pure proportional, threshold proportional, prized proportional, etc.) in order to minimize the differences of the distribution of power on the votes and on the seats. A more challenging proposal is to design an apportionment rule that, starting with a fixed number of seats (or better with a range for the number of seats), assigns the seats to the parties in order to minimize the difference in the two power distributions. We may go further, supposing to assign a different weight to the members of the different parties in order to have the same distribution of power on the votes and on the seats. A similar idea was already used for designing the VAP (Voting A Posteriori) system (see [5]), where the different weights of the votes were introduced to increase the governability of the Parliament.

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