Criteria for Assessing Voting and Electoral Systems

Hannu Nurmi

Public Choice Research Centre
University of Turku

Game Theory and Models of Voting

- Voting rules are instruments with many properties
- Some of these can be viewed as rationality properties
- Some are mutually compatible, some incompatible
- Patching existing rules may lead to new problems
- Some counterexamples are harder to come by than others
- This pertains the relevance of (negative) results

- Aggregating opinions.
- Making collective choices.
- Making individual choices
- Settling disagreements.
- Searching for consensus.

Rules make a difference

4 voters	3 voters	2 voters
A	E	D
B	D	C
C	B	B
D	C	E
E	A	A

5 options, 5 winners

Basic views:

- Arrovian view: collective opinions should be similar to the individual ones
- Condorcet requirements
- Consistency
- Choice set invariance
- Monotonicity

Borda's paradox

4 voters	3 voters	2 voters
A	B	C
B	C	B
C	A	A

Borda's points:

- plurality voting results in a bad outcome
- a superior system exists (Borda Count)

How does it work? Compute Borda scores and eliminate all candidates with no more than average score. Repeat until the winner is found.

Properties:

- Guarantees Condorcet consistency
- Is nonmonotonic

30	21	20	12	12	5
C	B	A	B	A	A
A	D	B	A	C	C
D	C	D	C	B	D
B	A	C	D	D	B

The Borda ranking: $A \succ C \succ B \succ D$ with D's score 97 being the only one that does not exceed the average of 150. Recomputing the scores for A, B and C, results in both B and C failing to reach the average of 100. Thus, A wins. Suppose now that those 12 voters who had the ranking $B \succ A \succ C \succ D$ improve A's position, i.e. rank it first, *ceteris paribus*. Now, both B and D are deleted and the winner is C.

Properties:

- Does not elect Condorcet losers
- Is nonmonotonic

6 voters	5 voters	4 voters	2 voters
A	C	B	B
B	A	C	A
C	B	A	C

How does it work? Pick the Condorcet winner. If none exists, choose the Borda winner.

Properties:

- Satisfies Cordorcet criteria
- Is monotonic
- Is inconsistent

4 voters	3 voters	3 voters	2 voters	2 voters
A	B	A	B	C
B	C	B	C	A
C	A	C	A	B

Voting system	**Criterion**								
	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
Amendment	1	1	1	1	0	0	0	0	0
Copeland	1	1	1	1	1	0	0	0	0
Dodgson	1	0	1	0	1	0	0	0	0
Maximin	1	0	1	1	1	0	0	0	0
Kemeny	1	1	1	1	1	0	0	0	0
Plurality	0	0	1	1	1	1	0	0	1
Borda	0	1	0	1	1	1	0	0	1
Approval	0	0	0	1	0	1	1	0	1
Black	1	1	1	1	1	0	0	0	0
Pl. runoff	0	1	1	0	1	0	0	0	0
Nanson	1	1	1	0	1	0	0	0	0
Hare	0	1	1	0	1	0	0	0	0

- a: the Condorcet winner criterion
- b: the Condorcet loser criterion
- c: the strong Condorcet criterion
- d: monotonicity
- e: Pareto
- f: consistency
- g: Chernoff property
- h: independence of irrelevant alternatives
- i: invulnerability to the no-show paradox

Examples:

- Arrow
- Gibbard-Satterthwaite
- Moulin
- Young

Consider a partition of a set N of individuals with preference
profile ϕ into two separate sets of individuals N_1 and N_2 with
corresponding profiles ϕ_1 and ϕ_2 over A and assume that
$f(\phi_1 \cap \phi_2) \neq \emptyset$. The social choice function f is consistent iff
$f(\phi_1 \cap \phi_2) = f(\phi)$, for all partitionings of the set of
individuals.
The same definition can be applied to social preference
functions. F is consistent iff $F(\phi_1) \cap F(\phi_2) \neq \emptyset$ implies that
$F(\phi_1) \cap F(\phi_2) = F(\phi)$.
As a choice function Kemeny's rule is inconsistent
(Fishburn). As a preference function it is consistent.

Ostrogorski's paradox:

issue	issue 1	issue 2	issue 3	the voter votes for
voter A	X	X	Y	X
voter B	X	Y	X	X
voter C	Y	X	X	X
voter D	Y	Y	Y	Y
voter E	Y	Y	Y	Y
winner	Y	Y	Y	?

- criterion A: relevant educational background
- criterion B: political experience
- criterion C: negotiation skills
- criterion D: substance expertise
- criterion E: relevant political connections

Suppose that the criterion-wise preference is formed on the basis of which alternative is better on more issues than the other. If all issues and criteria are deemed importance, the decision of which candidate the individual should vote is ambiguous: the row-column aggregation with the majority principle suggests X, but the column-row aggregation with the same principle yields Y.

Example

Nermuth. One of two competitors, X, is located at the following distance from the voter's ideal point in a multi-dimensional space. The score of X on each criterion is simply the arithmetic mean of its distances rounded to the nearest integer and in the case of a tie down to the nearest integer.

issue	1	2	3	4	average	score
criterion 1	1	1	2	2	1.5	1
criterion 2	1	1	2	2	1.5	1
criterion 3	1	1	2	2	1.5	1
criterion 4	2	2	3	3	2.5	2
criterion 5	2	2	3	3	2.5	2

Example

X's competitor Y, in turn, is located in the space as follows.

issue	1	2	3	4	average	score
criterion 1	1	1	1	1	1.0	1
criterion 2	1	1	1	1	1.0	1
criterion 3	1	1	2	3	1.75	2
criterion 4	1	1	2	3	1.75	2
criterion 5	1	2	1	2	1.75	2

Anscombe's paradox

Example

issue	issue 1	issue 2	issue 3
voter 1	Y	Y	X
voter 2	X	X	X
voter 3	X	Y	Y
voter 4	Y	X	Y
voter 5	Y	X	Y

Ostrogorski vs. Anscombe

Example

voter	issue 1	issue 2	issue 3	majority alternative
1	X	X	Y	X
2	X	Y	X	X
3	Y	X	X	X
4	Y	Y	Y	Y
5	Y	Y	X	Y

Cohen and Nagel (1934):

Example

death rate per 100.000	New York	Richmond
sub-population 1	179	162
sub-population 2	560	332
total death rate	187	226

System choice in simple settings

1. A satisfies the criterion, while B doesn't, i.e. there are profiles where B violates the criterion, but such profiles do not exist for B.

2. in every profile where A violates the criterion, also B does, but not vice versa.

3. in *practically all profiles* where A violates the criterion, also B does, but not vice versa ("A dominates B almost everywhere").

4. in a plausible probability model B violates the criterion with higher probability than A.

5. in those political cultures that we are interested in, B violates the criterion with higher frequency than A.

- impartial culture: each ranking is drawn from uniform probability distribution over all rankings
- impartial anonymous culture: all profiles (i.e. distributions of voters over preference rankings) equally likely
- unipolar cultures
- bipolar cultures

- cultures make a difference (Condorcet cycles, Condorcet efficiencies, discrepancies of choices)
- none of the cultures mimics "reality"
- IC is useful in studying the proximity of intuitions underlying various procedures

The fact that they involve intuitively plausible, "natural" or "obvious" desiderata. The more plausible etc. the more dramatic is the incompatibility.

Theorem

Moulin, Pérez: all Condorcet extensions are vulnerable to the no-show paradox.

Example

26%	47%	2%	25%
A	B	B	C
B	C	C	A
C	A	A	B

Black' procedure is vulnerable to the no-show paradox, indeed, to the strong version thereof.

1 voter	1 voter	1 voter	1 voter	1 voter
D	E	C	D	E
E	A	D	E	B
A	C	E	B	A
B	B	A	C	D
C	D	B	A	C

Here D is the Condorcet winner and, hence, is elected by Black.

Suppose now that the right-most voter abstains. Then the Condorcet winner disappears and E emerges as the Borda winner. It is thus elected by Black. E is the first-ranked alternative of the abstainer.

5 voters	5 voters	6 voters	1 voter	2 voters
A	B	C	C	C
B	C	A	B	B
D	D	D	A	D
C	A	B	D	A

Here Nanson's method results in B.

If one of the right-most two voters abstain, C – their favorite – wins. Again the strong version of no-show paradox appears.

The twin paradox occurs whenever a voter is better off if one or several individuals, with identical preferences to those of the voter, abstain. Here we have an instance of the twin paradox as well: if there is only one CBDA voter, C wins. If he is joined by another, B wins.

Starting profile:

7 voters	4 voters
A	B
B	C
C	A

Add a Condorcet paradox profile:

4 voters	4 voters	4 voters
A	B	C
C	A	B
B	C	A

to get a new Condorcet winner.

Some proofs are (almost) constructive, i.e. tell us how to generate paradoxes. Pérez uses the following auxiliary result. Let $p(x, y) =$ the no. of voters preferring x to y.

Theorem

For any Condorcet extension which is invulnerable to no-show paradox, for any situation (X, p) and for any pair x, z of alternatives, if $p(x, z) < min_{y \in X} p(z, y)$, then $x \notin f(X, p)$.

In words, the antecedence says that the minimum support for z is larger than the no. of votes x receives in comparison with z. The consequence says that then x is not elected (provided that the f is Condorcet and invulnerable).

Learning . . ., cont'd

The theorem is then used to construct an example.

5	4	3	3
t	y	x	x
y	z	t	t
z	x	z	y
x	t	y	z

Applying the Theorem to pairs $(z, y), (\mathbf{y}, \mathbf{t}), (t, x)$ it turns out that only x is chosen.

Add now 4 voters with ranking *zxyt* and apply Theorem to pairs $(t, x), (\mathbf{x}, \mathbf{z}), (z, y)$ to find that y is chosen.

Possible consensus states:

- consensus about everything, i.e. first, second, etc.
- consensus about the winner
- majority consensus about first rank
- majority consensus about Condorcet winner
- . . .

Possible distance measures:

- inversion metric (Kemeny)
- discrete metric

Upshot

We have (hopefully) seen that:

- system-criterion pairs give "asymmetric" information
- only important criteria ought to be focused upon
- the likelihood of encountering problems varies with the culture
- some counterexamples are much harder to find than others

What is called for is (much) more work on structural properties of problematic profiles.

Criteria for
assessment

The main
points of the
presentation

What are
voting rules
used for

Rationality of
rules

Improving old
systems

Varieties of
goodness

Majority rule
and
democracy

Principles of
system choice

How often are
the criteria
violated?

The no-show

📕 H. Nurmi.
Comparing Voting Systems.
Dordrecht: D. Reidel, 1987.

📕 H. Nurmi.
Voting Procedures under Uncertainty.
Berlin: Springer, 2002.

📕 D. Saari.
Basic Geometry of Voting.
Berlin: Springer 1995.

📕 D. Saari.
Decision and Elections.
Cambridge: Cambridge University Press, 2001.

📄 P. Fishburn.
Condorcet social choice functions.
SIAM Journal of Applied Mathematics, 33, 1977,
469-489.

📄 H. Moulin.
Condorcet's principle implies the no show paradox.
Journal of Economic Theory 45, 1988, 53-64.

📄 J. Pérez.
The strong no show paradoxes are common flaw in
Condorcet voting correspondences.
Social Choice and Welfare 18,2001, 601-616.