Centro di Cultura Scientifica "Alessandro Volta"

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UNIVERSITAS STUDIORUM INSUBRIAE

Indices for Parliaments

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Contents

1 The *FP* family of power indices for voting games

(see Fragnelli et al., 2009)

Power indices $\rightarrow$ measure of the influence of each member of a coalition on decisions
In the last decade $\rightarrow$ increasing attention in political science:

- study the voting power among EU states
- analyze the effects of institutional reforms

See Felsenthal and Machover (1997), Nurmi (1997 and 2000), Nurmi and Meskanen (1999), Dowding (2000), Aleskerov et al. (2002)

- For policy makers:
  Provide a power index with predictive value, designing a family of power indices that may be tailored on different situations with a suitable setting of some parameters
- For game theorists:
  Take into account the role of a party in favoring the formation of a majority, even if its influence on the majority in terms of seats (or percentage of votes) is null
  See Moretti and Patrone (2008) and related comments

The Archetype

A suitable analysis of the ideologies allows to represent the parties on a left-right axis $\rightarrow$ ordering of the parties

$$\Downarrow$$

contiguous winning coalition S: for all $i, j \in S$ if there exists $k \in N$ with $i < k < j$ then $k \in S$

- Let $W^c = \{S_1, S_2, ..., S_m\}$ be the set of contiguous winning coalitions
- Equal sharing of the unitary power among coalitions $S_j \in W^c$
- Equal sharing of the quota assigned to coalition S_j among its members
- Each player $i \in N$ sums up all the amounts received in the coalitions he belongs to

$$FP_i = \sum_{S_j \in W^c, S_j \ni i} \frac{1}{m} \frac{1}{s_j}, \qquad i \in N$$

where s_j is the cardinality of S_j

1.1 The Plausibility Criterion

- Set of winning coalitions
- Probability of a coalition to form
- Sharing rule inside each winning coalition

Set of winning coalitions

- *office-seeking*: size of the coalition
  - minimal winning coalition (MWC) - von Neumann and Morgenstern (1953)
  - minimal number coalition (MNC): minimum of the parties - Leiserson (1966)
  - minimal size coalition (MSC): minimum of the seats - Riker (1962)
- *policy-seeking*: ideological connection - Axelrod (1970), De Swaan (1973), Aleskerov (2008)
- other

Probability of a coalition to form

Equiprobability of coalitions may be not appealing

- ideological distance between the extreme parties - Bilal et al. (2001)
- distance between the preferred point of each party and the position of the coalition - Bilal et al. (2001)
- number of parties (proxy of ideological distance)
- number of parties or number of seats (cost of coordination and agreement): the larger the number, the lower the probability
- estimation on historical data - Heard and Swartz (1998)

Sharing rule inside each winning coalition

Equal sharing rule may be not suitable

- number of seats of each party
- critical role of parties (distance from potential opposition)
- median parties within the coalition - Bäck (2001)

1.2 Examples and Comparisons

Example 1 *Let [9; 6, 2, 6] be a weighted majority situation*
A unique contiguous winning coalition: $\{1, 2, 3\}$

Equally sharing the power among the players:

$$FP = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$\varphi = \delta = (\frac{1}{2}, 0, \frac{1}{2})$

Example 2 *Let [7; 1, 5, 2, 3] be a weighted majority situation*
Four contiguous winning coalitions: $\{2,3\},\{1,2,3\},\{2,3,4\},\{1,2,3,4\}$

$$FP = (0.146, 0.354, 0.354, 0.146)$$

Proportional sharing w.r.t. the number of seats

$$FP = (0.054, 0.573, 0.229, 0.143)$$

Sharing w.r.t. number of parties (probability = (0.353, 0.235, 0.235, 0.176)) and proportional sharing w.r.t. the number of seats

$$FP = (0.045, 0.597, 0.239, 0.119)$$

$\varphi = (0, 0.666, 0.167, 0.167), \delta = (0, 0.50, 0.25, 0.25)$

A unique minimal contiguous winning coalition: $\{2,3\}$

$$FP = (0, 0.50, 0.50, 0)$$

A player $i \in S$ is *contiguous-critical* for a contiguous winning coalition S if $S \setminus \{i\}$ is losing or is no longer contiguous

Example 3 *Let [24; 13, 8, 2, 10, 11] be a weighted majority situation*
Three contiguous winning coalitions: $\{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}$
Sharing vectors w.r.t. contiguous-critical parties: $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}), (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$

$$FP = \left(\frac{3}{36}, \frac{10}{36}, \frac{10}{36}, \frac{10}{36}, \frac{3}{36}\right)$$

Coherence in the choice of the parameters

- minimal size coalition $\leftrightarrow$ number of seats
- minimal number coalition $\leftrightarrow$ number of parties

1.3 Extension

(see Chessa, Fragnelli, 2011)

The hypothesis of contiguity of the parties can be relaxed, using a graph as in the cooperation structure by Myerson (1977)
Consider an undirected graph g whose vertices correspond to the agents and the edges connect pairs of agents that may communicate
Given a game (N, v), define the restricted game (N, v_g) *as* $v_g(S) = \sum_{T \in S/g} v(T), S \subseteq N$ *where S/g is the partition of S induced by the connected components of g*
A coalition is feasible and its worth is "effective" if the vertices associated to its players are connected, otherwise the worth of the coalition is the sum of the worths of the communicating subcoalitions
The Myerson value *of (N, v) is the Shapley value of (N, v_g)*

The central role played by the contiguous coalitions is now assigned to the connected ones
The model of the left-right axis is a coalition structure represented by a line-graph
This approach allows for a multidimensional space of the parties

2 Evaluation of a Parliament

(see Fragnelli et al, 2005)

The choice of the best electoral system for a Parliament is very hard:

- Too many variables involved ⇒ too difficult to balance all of them
- Complex methods are difficult to be understood and managed by voters

Some theorems - Arrow's (1950) and McKelvey's (1976) *in primis* - exclude the possibility of finding out the optimal rule, but no theorem prohibits finding out an *empirical* criterion of choice among two rules

Compare the performance of the two systems
with reference to the same set of real preferences of voters

Votes are affected by the electoral system in use

$\Downarrow$

How would you vote if the electoral system were X in a country where the system is Y?

2.1 The Simulative Approach

Simulation is widely used in electoral systems analysis

- examine a specific mixed-member suggestion (Brichta, 1991)
- assess the proportionality of Chilean electoral system via opinion polls (Valenzuela, Siavelis, 1991)
- analyze the contendibility of a two-party system (Bender, Haas, 1996)
- find out equilibria in multiparty spatial models (Lomborg, 1997)
- estimate the effect of the adoption of alternative vote in Canada (Bilodeau, 1999)
- assess the motivations of the electoral reform in Italy (Navarra, Sobbrio, 2001)

Referring to comparison of systems

- pioneering papers: Merrill (1984, 1985), Mueller (1989)
- analyze the effect in Italy of a change to a number of electoral systems (Gambarelli, Biella, 1992)
- compare six majoritarian systems but without reference to a Parliament (Christensen, 2003)

2.2 The Choice of the Optimal Electoral System

The choice may be affected by a lot of facets of the political process

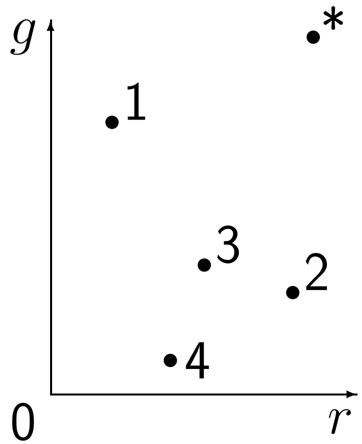
- *representativeness*, R: efficiency in representing electors' willingness
- *governability*, G: effect on the efficiency of the resulting government

In addition:

- incentives for politicians (Riker, 1982; Myerson, 1995)
- corruption (Myerson, 1993, 2001; Persson, Tabellini, Trebbi, 2001)
- information / participation of voters (Mudambi, Navarra, Nicosia, 1995; Mudambi, Navarra, Sobbrio, 1999)
- power of the lobbies (Myerson, 1995)
- strategic choices (Levin, Nalebuff, 1995)
- complexity of the voting system (Levin, Nalebuff, 1995)
- protection of the minorities (Levin, Nalebuff, 1995; Rae, 1995; Sen, 1995)
- risk of extreme choices (Levin, Nalebuff, 1995)
- use of votes as a "voice" device (Sen, 1995; Brennan, Hamlin, 1998)
- public spending (Persson, Tabellini, 1998, 2001; Milesi-Ferretti, Perotti, Rostagno, 2000)
- overall welfare (Mueller, Stratmann, 2000)
- responsiveness of the government's choice to the preferences of the voters (Shugart, 2001)
- ...

2.3 The Choice with Two Parameters

Dominance



System "*" is *dominant* (it is very likely not to exist)

System 4 is *dominated* (it may safely be excluded)

Systems 1, 2, 3 are *alternative systems* (Pareto optimality → which system should be chosen?)

Alternative systems may be compared via a social utility function (*SUF*), e.g. a Cobb-Douglas function

$$U = Kg^a r^b$$

where K is a suitable constant

a and b are the partial elasticity of U w.r.t. g and r

$$X \succ Y \iff Kg_X^a r_X^b > Kg_Y^a r_Y^b$$

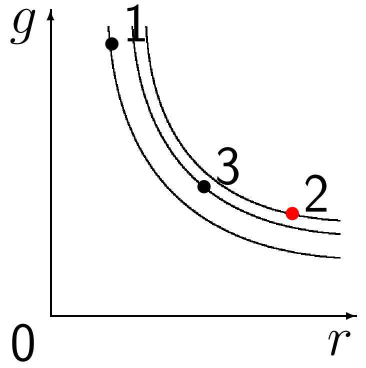
Let $p = \dfrac{a}{b}$

$$X \succ Y \iff \left(\frac{g_X}{g_Y}\right)^{pb} > \left(\frac{r_Y}{r_X}\right)^{b}$$

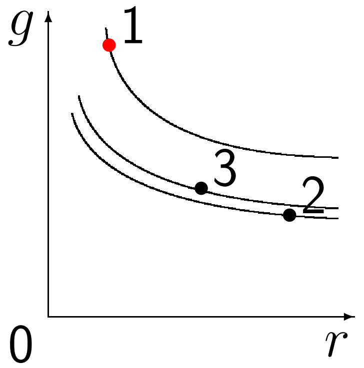
Remark 1 *p may be characterized as the price in terms of a relative variation of r that the community accepts to pay for a given relative opposite variation of g*
$p = 2$ means that it is worthwhile to accept a 20% reduction of r to gain a 10% increase of g

Graphically - Indifference curves

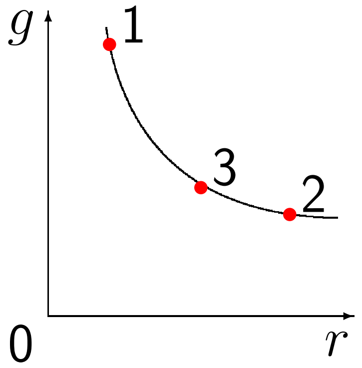
$$r = \frac{\left(\frac{U^*}{K}\right)^{\frac{1}{b}}}{g^{\frac{a}{b}}}$$



but ...



or ...



the three systems are indifferent

3 Disproportionality Indices

This list is taken from Karpov (2008)
Representativeness and disproportionality are strongly related; for the first the greater the better, for the second the less the better

Notations

$N = \{1, ..., n\}$	set of parties
S_i, V_i	number of seats, votes of party $i \in N$
s_i, v_i	percentage of seats, votes of party $i \in N$

3.1 Absolute Deviation Indices

Maximum Deviation Index

$$MD = \max_{i \in N} |s_i - v_i|$$

Rae Index

$$I_{Rae} = \frac{1}{n} \sum_{i \in N} |s_i - v_i|$$

Loosmore-Hanby Index

$$I_{LH} = \frac{1}{2} \sum_{i \in N} |s_i - v_i|$$

Grofman Index

$$I_{Gr} = \frac{1}{E} \sum_{i \in N} |s_i - v_i|$$

where $E = \frac{1}{\sum_{i \in N} v_i^2}$ if the effective number of parties

Lijphart Index

$$I_L = \frac{|s_i - v_i| + |s_j - v_j|}{2}$$

where i and j are the two largest parties

3.2 Quadratic Indices

Gallagher Index

$$Lsq = \sqrt{\frac{1}{2}\sum_{i\in N}(s_i - v_i)^2}$$

Variations

$$H_k = \sqrt[k]{\frac{1}{k}\sum_{i\in N}(s_i - v_i)^k}$$

$$\tilde{H}_k = \sqrt[k]{\frac{1}{2}\sum_{i\in N}(s_i - v_i)^k}$$

Monroe Index

$$I_M = \sqrt{\frac{\sum_{i\in N}(s_i - v_i)^2}{1 + \sum_{i\in N} v_i^2}}$$

Gatev Index

$$I_{Ga} = \sqrt{\frac{\sum_{i\in N}(s_i - v_i)^2}{\sum_{i\in N}(s_i^2 + v_i^2)}}$$

Ryabtsev Index

$$I_R = \sqrt{\frac{\sum_{i\in N}(s_i - v_i)^2}{\sum_{i\in N}(s_i + v_i)^2}}$$

Szalai Index

$$I_S = \sqrt{\frac{\sum_{i\in N}\left(\frac{s_i - v_i}{s_i + v_i}\right)^2}{n}}$$

Variation

$$\tilde{I}_S = \sqrt{\frac{1}{2}\sum_{i\in N}\frac{(s_i - v_i)^2}{s_i + v_i}}$$

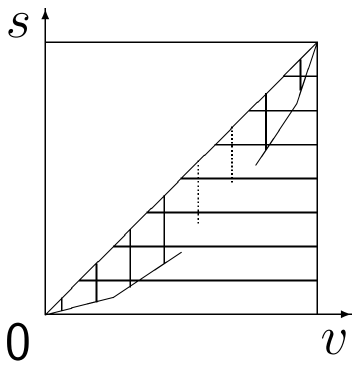
3.3 Aleskerov-Platonov Index

$$R = \frac{1}{k} \sum_{i \in O} \frac{s_i}{v_i}$$

where $O = \{i_1, ..., i_k\}$ is the set of overrepresented parties.

3.4 Inequality Indices

Gini Index



$$\frac{s_1}{v_1} \leq \frac{s_2}{v_2} \leq \ldots \leq \frac{s_{n-1}}{v_{n-1}} \leq \frac{s_n}{v_n}$$

G is the ratio of the areas

Atkinson Index

$$A = 1 - \left[\sum_{i \in N} v_i \left(\frac{y_i}{\mu} \right)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}$$

where $y_i = \frac{S_i}{V_i}$, $\mu = \frac{S}{V}$ and ε is a measure of inequality; equivalently

$$A = 1 - \left[\sum_{i \in N} v_i \left(\frac{s_i}{v_i} \right)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}$$

Generalized Entropy

$$GE = \frac{1}{\alpha^2 - \alpha} \left[\sum_{i \in N} v_i \left(\frac{y_i}{\mu} \right)^{\alpha} - 1 \right]$$

where α is a parameter; equivalently

$$GE = \frac{1}{\alpha^2 - \alpha} \left[\sum_{i \in N} v_i \left(\frac{s_i}{v_i} \right)^{\alpha} - 1 \right]$$

3.5 Objective Function Indices

Sainte-Lague Index

$$SL = \sum_{i \in N} v_i \left(\frac{s_i}{v_i} - 1 \right)^2$$

D'Hondt Index

$$H = \max_{i \in N} \frac{s_i}{v_i}$$

3.6 Three More Indices and the Issue of Power

Ortona index (see Fragnelli-Monella-Ortona, 2005)
It is based on the difference between seats assigned by a given electoral system and seats assigned by a perfect proportional system, PP, i.e. supposing a unique nation-wide proportional district and assigning the rest to the largest decimals, avoiding to combine votes and seats

$$d = \frac{\sum_{i \in N} |S_i - S_i^{PP}|}{\sum_{i \in N} |S_i^u - S_i^{PP}|}$$

where N is the set of parties, S_i is the number of seats of party i with the system under consideration, S_i^{PP} is the number of seats of party i with the perfect proportional system and S_i^u is the total number of seats for the relative majority party and 0 otherwise

Fragnelli index (see Fragnelli, 2009)
Starting from the vote share $(v_1, \ldots, v_n)$ and the seat share $(s_1, \ldots, s_n)$, define two simple games (N, w) and (N, u), where $N = \{1, \ldots, n\}$ is the set of parties and the characteristic functions w and u are defined using the vote share and the seat share, respectively
This disproportionality index measures the distance of the distribution of power on the votes and on the seats, i.e. $\sum_{i \in N} |\varphi_i(w) - \varphi_i(u)|$

Remark 2 *The disproportionality is minimal, i.e. equal to 0, when the power of each party is identical in the two distributions*
A particular case happens when the percentages of votes coincide with the percentages of seats

The index may assume a value larger than 1

Example 4 *Consider a relative majority system with two parties P_1 and P_2 and three districts A, B and C, each one electing a single member*
The results of the elections are in the following table:

Parties	A	B	C	$\%$ of votes	$\#$ of seats
P_1	16	6	8	60	1
P_2	2	8	10	40	2

In this case $\varphi(w) = (1, 0)$ and $\varphi(u) = (0, 1)$, so the index is $1 + 1 = 2$

Normalize the index in the interval $[0, 1]$, simply dividing by 2, as in the worst case the two distributions of power may assign complementary values:

$$d^{\Omega} = \frac{\sum_{i \in N} |\varphi_i(w) - \varphi_i(u)|}{2}$$

Gambarelli-Biella index (see Gambarelli-Biella, 1992)

It combines the traditional approach, which considers vote and seat shares, with the idea of measuring the distance of the distributions of power related to the votes and to the seats:

$$\Delta = \max_{i \in N} \{|v_i - s_i|, |\varphi_i(w) - \varphi_i(u)|\}$$

Remark 3 *Note that d^{Ω} is based on norm 1, while Δ is based on norm ∞, so it is possible to define other indices based on other norms*

For computing the Fragnelli and Gambarelli-Biella indices two of the most popular power indices are used:

- the Shapley-Shubik index (1954)

$$\phi_i(w) = \sum_{S \in W, S \ni i} \frac{(|S|-1)!(n-|S|)!}{n!}[w(S) - w(S \setminus \{i\})] \quad \forall i \in N$$

  where W is the set of winning coalitions

- the Public Good Index (Holler, 1982)

$$PGI_i(w) = \frac{h_i}{\sum_{k \in N} h_k} \quad \forall i \in N$$

  where h_i is the number of minimal winning coalitions including party $i \in N$

The former has the monotonicity property, so that it assigns larger power to parties with greater number of seats, while the latter does not respect monotonicity and the differences of power are often very small w.r.t other non-monotonic indices

Case-study 1

Three parties A, B, C receive a percentage of votes of $49.5\%, 48.5\%, 2.0\%$, respectively and 6 seats have to be assigned. Consider three different seats assignments:

	A	B	C
s_{PP}	3	3	0
s	3	2	1
s_{PO}	2	2	2

s_{PP} assigns the seats according to the perfect proportional system; s guarantees one seat to party C, so party A receives one seat more than party B; finally, s_{PO} (PO after Power Oriented) guarantees a distribution of power on the seat sharing equal to the distribution of power on the vote sharing

	MD	I_{Rae}	I_{LH}	I_{Gr}	I_L	Lsq	I_M	I_{Ga}	I_R	I_S	G	R	SL	H
s_{PP}	0.020	0.013	0.020	0.019	0.010	0.018	0.021	0.026	0.018	0.577	0.020	1.021	0.021	1.031
s	0.152	0.101	0.152	0.146	0.078	0.149	0.173	0.226	0.162	0.587	0.228	1.010	1.123	8.333
s_{PO}	0.313	0.209	0.313	0.301	0.157	0.271	0.315	0.425	0.315	0.598	0.313	16.667	5.009	16.667

s_{PO} is the worst under all the indices

Shapley-Shubik and Public Good indices:

	A	B	C
$\phi(v)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\phi(s_{PP})$	$\frac{1}{2}$	$\frac{1}{2}$	0
$\phi(s)$	$\frac{4}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$\phi(s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$PGI(v)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$PGI(s_{PP})$	$\frac{1}{2}$	$\frac{1}{2}$	0
$PGI(s)$	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$PGI(s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Added indices:

	d	d^{Ω}		Δ	
		ϕ	PGI	ϕ	PGI
s_{PP}	0	0.333	0.333	0.333	0.333
s	0.333	0.333	0.167	0.333	0.167
s_{PO}	0.667	0	0	0.313	0.313

The index by Ortona that does not account the power assigns the worst value to s_{PO}, that on the other hand receives the best score under the index by Fragnelli that is strongly power-oriented; the index by Gambarelli-Biella is influenced by the seat assignment

Case-study 2

Three parties A, B, C receive a percentage of votes of $49.9\%, 25.2\%, 24.9\%$, respectively; suppose two different Parliaments with 9 and 10 seats, respectively. Consider the assignment of the seats under PP (s_{PP}) and looking at the power distribution (s_{PO}):

	A	B	C
$9 - s_{PP}$	5	2	2
$9 - s_{PO}$	4	3	2
$10 - s_{PP}$	5	3	2
$10 - s_{PO}$	4	3	3

s_{PP} gives to party A the absolute majority in the Parliament with 9 seats and the veto power with 10 seats, differently from the distribution of votes and s_{PO}

	MD	I_{Rae}	I_{LH}	I_{Gr}	I_L	Lsq	I_M	I_{Ga}	I_R	I_S	G	R	SL	H
$9 - s_{PP}$	0.057	0.038	0.057	0.042	0.043	0.049	0.059	0.078	0.056	0.579	0.057	1.113	0.013	1.113
$9 - s_{PO}$	0.081	0.054	0.081	0.061	0.068	0.072	0.087	0.119	0.084	0.584	0.082	1.323	0.035	1.323
$10 - s_{PP}$	0.049	0.033	0.049	0.037	0.025	0.049	0.059	0.079	0.056	0.580	0.073	1.096	0.019	1.190
$10 - s_{PO}$	0.099	0.066	0.099	0.074	0.074	0.086	0.103	0.143	0.102	0.583	0.100	1.198	0.039	1.205

Again, s_{PO} is the worst under all the indices, with both 9 and 10 seats

Shapley-Shubik and Public Good indices:

	A	B	C
$\phi(v)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\phi(9-s_{PP})$	1	0	0
$\phi(9-s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\phi(10-s_{PP})$	$\frac{4}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$\phi(10-s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$PGI(v)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$PGI(9-s_{PP})$	1	0	0
$PGI(9-s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$PGI(10-s_{PP})$	$\frac{2}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$PGI(10-s_{PO})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Added indices:

	d	d^{Ω}		Δ	
		ϕ	PGI	ϕ	PGI
$9-s_{PP}$	0	0.667	0.667	0.667	0.667
$9-s_{PO}$	0.250	0	0	0.081	0.081
$10-s_{PP}$	0	0.333	0.167	0.333	0.167
$10-s_{PO}$	0.200	0	0	0.099	0.099

Again, the index by Ortona assigns the worst value to s_{PO}, that receive the best score under the two power-oriented indices

Remark 4

- *The issue of power allows new perspective for analyzing the disproportionality*
- *With a higher number of seats it is possible to reduce the differences among the indices For instance, referring to case-study 1 with 15 seats, s_{PP} and s correspond to $(8, 7, 0)$ and $(7, 7, 1)$, respectively, and the first is easily rejected because Party A would have the absolute majority*
  To guarantee the same distribution of power on the votes and on the seats it is sufficient to assign to each party a number of seats greater than or equal to 1 and less than or equal to 7, i.e. s_{PO} can be selected equal to s

4 Governability indices

Ortona index (see Fragnelli-Monella-Ortona, 2005)

$$g_h = \frac{1}{m_h + 1} + \frac{1}{m_h(m_h + 1)} \frac{f_h - \frac{T}{2}}{\frac{T}{2}}$$

where m_h is the number of parties of the governing coalition under system h that may destroy the majority if they withdraw, f_h is the number of seats of the majority under system h and T is the total number of seats in the Parliament

4.1 Power and Governability

(see Fragnelli, 2009)

Why not to use power indices also for governability?

Propensity to disrupt (Gately, 1974) for player $i \in N$:

$$\frac{x(N \setminus \{i\}) - v(N \setminus \{i\})}{x_i - v(i)}$$

Remark 5 *$x_i - v(i)$ and $x(S \setminus \{i\}) - v(S \setminus \{i\})$ may be used to evaluate the stability of a parliamentary coalition $S \subseteq N$, while the propensity to disrupt is usually equal to - 1*

When S represents a majority coalition, it is possible to emphasize its power using the Owen value (Owen, 1977), Ω, instead of the Shapley value, because it assigns null power to each player not in the majority

$$\Omega_i(K) = \sum_{\substack{H \subseteq K \\ T_j \notin H}} \sum_{\substack{S \subseteq T_j \\ i \notin S}} \frac{h!(k-h-1)!s!(t_j-s-1)!}{k!t_j!} [v(H \cup S \cup \{i\}) - v(H \cup S)], i \in N$$

where $K = \{T_1, ..., T_k\}$ is the *a priori unions* structure, $n = |N|, s = |S|, h = |H|, k = |K|$ and $t_j = |T_j|$

Consider $\Omega_i(S) - \Omega_i(S \setminus \{i\}), \quad i \in S$, where $\Omega(S)$ is the Owen value when the *a priori unions* structure considers the coalition S and all the other players as singletons, i.e. $K = \{S, \{i_1\}, ..., \{i_{n-s}\}\}$, where $s = |S|$

$\Omega_i(S) - \Omega_i(S \setminus \{i\})$ is a measure of the propensity of party i to stay in the majority

Consider $\Omega_i(S, N \setminus S) - \Omega_i(S \setminus \{i\}, (N \setminus S) \cup \{i\}), \quad i \in S$, where $\Omega(S, N \setminus S)$ is the Owen value when the *a priori unions* structure K_S considers the coalition S and the complementary coalition $N \setminus S$, i.e. $K = \{S, N \setminus S\}$

$\Omega_i(S, N \setminus S) - \Omega_i(S \setminus \{i\}, (N \setminus S) \cup \{i\})$ is a measure of the propensity of party i to stay in the majority

Consequently, the higher is the propensity of the parties to stay in the majority, the higher is the stability of the majority

Summing up on all the players in S, we get the following governability index $\sum_{i \in S}(\Omega_i(S) - \Omega_i(S \setminus \{i\})) = 1 - \sum_{i \in S} \Omega_i(S \setminus \{i\})$
(if S is a majority $\sum_{i \in S} \Omega_i(S) = 1$)

Summing up on all the players in S, we get the following governability index $\sum_{i \in S}(\Omega_i(S, N \setminus S) - \Omega_i(S \setminus \{i\}, (N \setminus S) \cup \{i\})) = 1 - \sum_{i \in S} \Omega_i(S \setminus \{i\}, (N \setminus S) \cup \{i\})$
(if S is a majority $\sum_{i \in S} \Omega_i(S) = 1$)

Remark 6 *According to standard literature, the governability is maximal, i.e. equal to 1, when* S *is such that each subcoalition* $S \setminus \{i\}$ *is winning, i.e. the majority is not affected whichever party leaves it. Note that for each party* $i \in S$, $\Omega_i(S \setminus \{i\}) = 0$ *and* $\Omega_i(S \setminus \{i\}, (N \setminus S) \cup \{i\}) = 0$

These indices may assume a negative value

Example 5 *Consider a majority game* (N, v) *in which only the grand coalition is winning*
In this case $\Omega_i(N) = \frac{1}{n}, i \in N$*; if party* k *leaves the majority its power becomes* $\Omega_k(N \setminus \{k\}) = \Omega_k(N \setminus \{k\}, \{k\}) = \frac{1}{2}$ *and the power of the other* $n-1$ *parties becomes* $\Omega_i(N \setminus \{k\}) = \Omega_i(N \setminus \{k\}, \{k\}) = \frac{1}{2}\frac{1}{n-1}, i \in N \setminus \{k\}$
So, both indices $1 - \sum_{i \in N} \Omega_i(N \setminus \{i\}) = 1 - \sum_{i \in N} \Omega_i(N \setminus \{i\}, \{i\}) = 1 - \sum_{i \in N} \frac{1}{2}$ *are negative when* $n \geq 3$

Normalize the index in the interval $[0, 1]$, simply dividing by n, as $0 \leq \Omega_i(S \setminus \{i\}) \leq 1$:

$$g^{\Omega} = 1 - \frac{\sum_{i \in S} \Omega_i(S \setminus \{i\})}{n}$$

Normalize the index in the interval $[0, 1]$, simply dividing by n, as $0 \leq \Omega_i(S \setminus \{i\}, (N \setminus S) \cup \{i\}) \leq 1$:

$$g^{\Omega^*} = 1 - \frac{\sum_{i \in S} \Omega_i(S \setminus \{i\}, (N \setminus S) \cup \{i\})}{n}$$

4.2 A Comparative Example

(see Fragnelli, Ortona, 2006)

Voting system	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
$PP(=v)$	18	22	5	4	0	**5**	0	**1**	0	**17**	**28**	0
$P-4$	19	22	5	**4**	0	**5**	0	0	0	**17**	**28**	0
$P(20)$	14	16	3	3	0	3	0	1	0	**12**	**48**	0
M	14	**36**	0	0	0	0	0	0	0	0	**50**	0
$2R$	14	**36**	0	0	0	0	0	0	0	0	**50**	0
C	**14**	**36**	0	0	0	0	0	0	0	**4**	46	0
B	**9**	**44**	0	0	0	0	0	0	0	15	32	0
A	**18**	**37**	0	0	0	0	0	0	0	18	27	0
$I-25$	15	33	1	**1**	0	**1**	0	0	0	**4**	**45**	0
$I-75$	17	25	4	3	0	**4**	0	**1**	0	**13**	**33**	0

PP	Pure Proportionality	M	Relative Majority	B	Borda Count
$P-n$	Threshold Proportionality	$2R$	Two-round Runoff	A	Approval Voting
$P(n)$	Prized Proportionality	C	Condorcet Method	$I-n$	Mixed-member

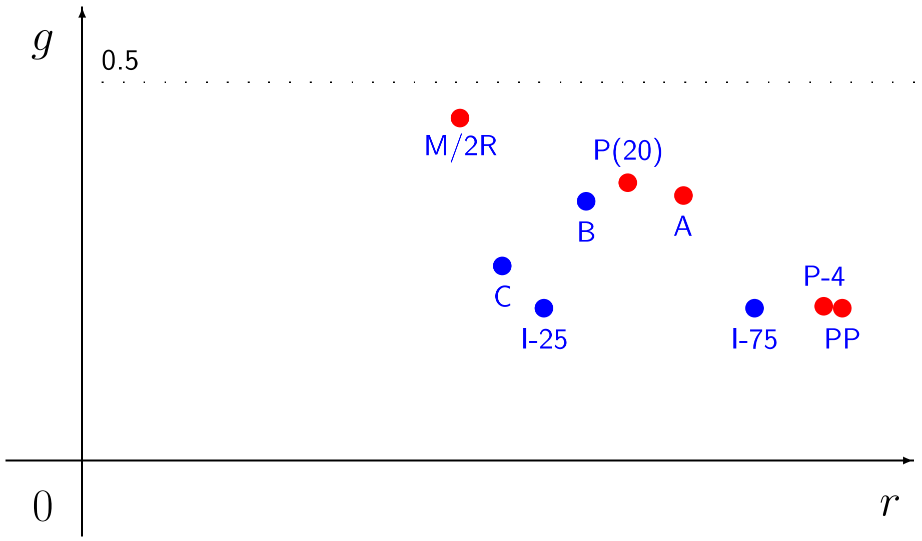
A unique 100-seat constituency for the PP and $P-n$, a unique (100 - n)-seat constituency for $P(n)$, a n-seat plus 100 - n one-seat constituencies for $I-n$ and 100 one-seat constituencies for all the other systems

Bold numbers identify the parties forming the majority in each system

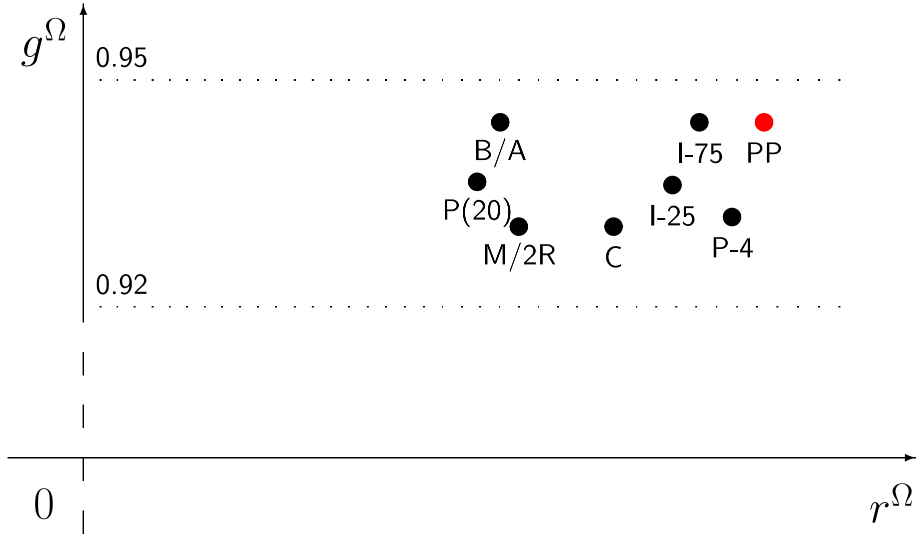
Indices:

$Voting\ system$	$r = 1 - d$	g	$r^\Omega = 1 - d^\Omega$	g^Ω	g^{Ω^*}
PP	1.000	0.201	1.000	0.944	0.895
$P - 4$	0.986	0.204	0.958	0.932	0.896
$P(20)$	0.722	0.367	0.621	0.937	0.917
M	0.500	0.453	0.676	0.931	0.917
$2R$	0.500	0.453	0.676	0.931	0.917
C	0.556	0.257	0.801	0.931	0.896
B	0.667	0.343	0.651	0.944	0.917
A	0.795	0.350	0.651	0.944	0.917
$I - 25$	0.611	0.201	0.879	0.936	0.896
$I - 75$	0.889	0.201	0.915	0.944	0.897

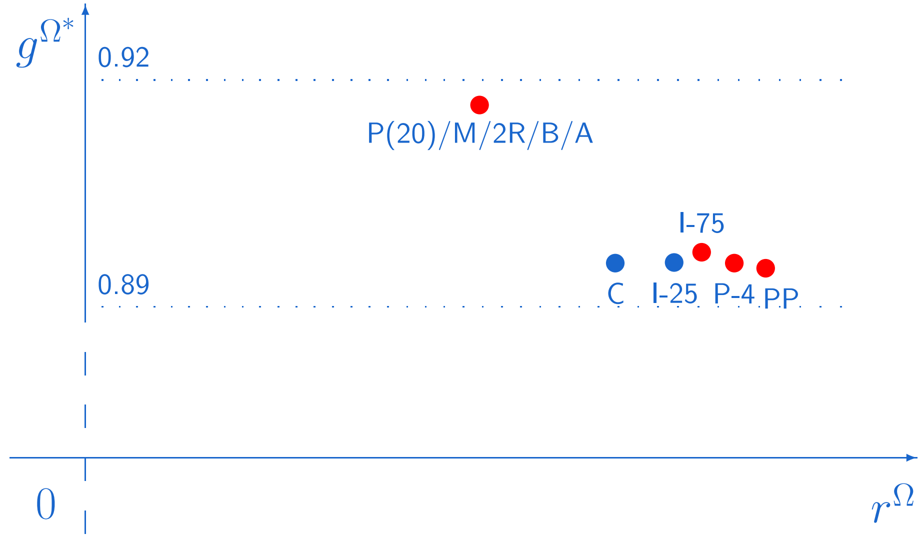
Graphical representation (different vertical scale):



In this case r and g produce six Pareto optimal systems



In this case r^{Ω} and g^{Ω} identify PP as a dominant system



In this case r^{Ω} and $g^{\Omega *}$ produce eight Pareto optimal systems

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Thank you for your attention!