On Axiomatization of Power Index of Veto

Jacek Mercik

Wroclaw University of Technology, Wroclaw, Poland
jacek.mercik@pwr.wroc.pl

Abstract. Relations between all constitutional and government organs must be moderated and evaluated depending on their way of decision making. Among their attributes one may find the right to veto. It is known already that a priori veto is rather strengthening the position of beholder. The evaluation of a power to make a decision is directly connected with a way of power measuring, i.e. with power index choice. In the paper we consider axiomatic base for such choice of an index of power evaluation.

Keywords: veto, power index, axioms.

1 Introduction

Relations between all constitutional and government organs must be moderated and evaluated depending on their way of decision making. Among different attributes of collective decision making one may find the right to veto. We think that veto a priori is rather strengthening the position of beholder. So, any considerations about consensus process must include evaluation of veto attribute as well, for to preserve the balance between sides.

The main goal of the paper is the analysis of axioms of power measure connected with veto attribute of a decision maker. In certain cases, it is possible to calculate a value of power of veto attributed to the decision maker and to give the exact value of the power index as well. In other cases, it is only possible to compare the situation with and without veto attribute. However, significant numbers of power indices are in use for evaluation of power of player with special emphasizing done for power of veto (for example: Bertini *et al.* 2012, Chessa, Fragnelli 2012, Mercik 2009, 2011), but there is no convincing arguments for choice of one or another power index. The main differences between these indices are the ways in which coalition members share the final outcome of their cooperation and the kind of coalition players choose to form. In this paper we would like to examine the base of such choice in axiomatic way.

adfa, p. 1, 2011.

2 Example of veto game

In Poland all bills are resolved if:

- An absolute majority of representatives and the president are for[1], or
- In the case of a veto by the president, at least 3/5 of the representatives are for[2].

In the case of Polish parliamentary bills acceptation process we may see that generally there is cooperative game schema where the president, the *Sejm* and the Senate must form a coalition for to accept a bill. Therefore a given power index may be in use directly to evaluate their influence on the legislation process with an idea that greater value of the index represents greater influence on the process. The overview of possible indices one may find for example in Hołubiec and Mercik (1994).

3 Basic notions.

Let N = {1,2,…,n} be the set of players. A game on N is given by a map $v: 2^N \to R$ with $v(\emptyset) = 0$. The space of all games on N is denoted by G. A coalition $T \in 2^N$ is called a carrier of v if $v(S) = v(S \cap T)$ for any $S \in 2^N$.

The domain $SG \subset G$ of *simple games* on *N* consists of all $v \in G$ such that

(i) $v(S) \in \{0,1\}$ for all $S \in 2^N$;

(ii) $v(N) = 1$;

(iii) v is monotonic, i.e. if $S \subset T$ ***then*** $v(S) \le v(T)$.

A coalition S is said to be winning in $v \in SG$ if $v(S) = 1$, and losing otherwise. Therefore, the voting upon a bill is equivalent to formation of a winning coalition consists of voters.

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[1] One may notice that the Polish *Senate* has no effective influence during the legislative process. The *Sejm* may reject the objections of the Senate at any moment by a simple majority, i.e. 231 deputies when all of them are present (460). Usually in the a priori analysis we only consider simple majority winning coalitions.

[2] This is a slightly simplified model, because the Supreme Court may also by simple majority recognize the bill as contradicting the Constitutional Act (or both chambers may change the Constitutional Act itself).

A simple game *(N, v)* is said to be proper, if and only if it is satisfied that for all $T \subset N$, if *v(T)=1* then *v(N\T)*=0.

Consider a simple game *(N, v)* and a coalition $S \subset N$. *(N, v)* is said to be **S-**unanimous, if and only if it is satisfied that v(T)=1 if and only if $T \supset S$.

4 The sense of veto

The meaning of veto can be explained by the following artificial example: $\{2; 1_a, 1_b, 1_c\}$ where the voting is a majority voting (voting quota equals 2) and weights of all voters *a, b, c* (*N=3*) are equal and fixed at 1. As it can be seen, there are four winning coalitions: {a,b}, {a,c}, {b,c}, {a,b,c}. The first three coalitions are vulnerable and the veto (called the veto of the first degree (Mercik, 2011)) of any coalition's members transforms it from winning into non-winning one. The classical example of such a veto is a possible veto of permanent members of the Security Council of the United Nations.

The last coalition, {a,b,c}, is different: a single member's veto can be overruled by two other members. This type of veto is called the second degree veto. A very typical example of such a veto is a presidential veto (at least in Poland or USA, for example), which under certain circumstances can be overruled.

A coalition structure $P = \{P_1, P_2, \dots, P_m\}$ over *N* is a partition of *N*, that is $\bigcup_{k=1}^{m} P_k = N$ and $P_k \cap P_h = \emptyset$ when $k \neq h$. A coalition structure with veto $Pv = \{P_1, \dots, \{j\}, \dots, P_m\}$ over *N* for *j=1, m* is a coalition structure P where at least one union is a singleton and at least one of the singletons is attributed with veto. The veto can be of the first or the second degree type.

The example.

One possible partition of Security Council of the UN's members:

$$P = \{\{P_1\}, \{P_2\}, \{P_3\}, \{P_4\}, \{P_5\}, \{NP_6, \dots, NP_{15}\}\},$$

where each permanent member P_i has veto attribute and the rest of SC's members create a coalition. Of course, different combinations of partitions are also possible.

A power index is a mapping $\varphi : SG \rightarrow R^n$. For each $i \in N$ and $v \in SG$, the i^{th} coordinate of $\varphi(v) \in R^n$, $\varphi(v)(i)$, is interpreted as the voting power of player *i* in the game *v*. In the literature there are two dominating power indices: the Shapley-Shubik power index and the

Banzhaf power index. Both base on the Shapley value concept[3]. The Shapley (1953) value is the value $\varphi : G \rightarrow R^n$, $v \rightarrow$ (φ_1 (v),φ_2(v),…,φ_n(v)) where for all $i \in N$

$$\varphi_i^{SS}(v) = \sum_{S \subset N, i \notin S} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)] \quad (1)$$

The Shapley-Shubik power index for simple game (Shapley, Shubik 1954) is the value $\varphi : SG \rightarrow R^n$, $v \rightarrow$(φ_1 (v),φ_2(v),…,φ_n(v)), where for all $i \in N$

$$\varphi_i^{SS}(v) = \sum_{S \subset N, i \notin S} \frac{s!(n-s-1)!}{n!} \quad (2)$$

The Banzhaf power index (Banzhaf, 1965) for simple game[4] is the value $\varphi : SG \rightarrow R^n$, $v \rightarrow$(φ_1 (v),φ_2(v),…,φ_n(v)) where for all $i \in N$

$$\varphi_i^{B}(v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N\{i\}} [v(S \cup \{i\}) - v(S)] \quad (\text{Errore. Il segnalibro non è definito.})$$

If one applies a partition structure P then Shapley-Shubik power index may be defined as following (Alonso-Meijide *et al*. 2007):

$$\varphi_i^{SS}(v,P) = \sum_{R \subseteq M \setminus \{k\}} \sum_{T \subseteq P_k \setminus \{i\}} \frac{(r+t)!(m+p_k-r-t-2)!}{(m+p_k-1)!} [v(Q \cup T \cup \{i\}) - v(Q \cup T)] \quad (\text{Errore. Il segnalibro non è definito.})$$

for all $i \in N$ and all (N, v, P) being a game with partition structure, where $P_k \in P$ is the union such that $i \in P_k$, $Q = \cup_{r \in R} P_r$.

An analogue definition of the Banzhaf power index for a game with partition structure can be formulated as following (Owen, 1982):

$$\varphi_i^{B}(v,P) = \sum_{R \subseteq M \setminus \{k\}} \sum_{T \subseteq P_k \setminus \{i\}} \frac{1}{2^{m-1}} \frac{1}{2^{p_k-1}} [v(Q \cup T \cup \{i\}) - v(Q \cup T)] \quad (3)$$

[3] The overview of the discussion about both power indices one may find in Laruelle, Valenciano (2000), Turnovec *et al.* (2004, 2008).

[4] This power index is called also as Banzhaf-Penrose power index. The Penrose's work from 1946 presented an analogue attempt to the concept of power for simple games.

for all $i \in N$ and all (N, v, P) being a game with partition structure, where $P_k \in P$ is the union such that $i \in P_k$, $m = \| M \|$, $p_k = \| P_k \|$ and $Q = \cup_{r \in R} P_r$.

Both power indices formulated for games with partition structure give us the opportunity to represent such decisive bodies as parliaments, parliament-president or so. Especially in a parliament the partition structure is evident when party system and block voting are observed.

As we may conclude at this stage of the proceeding the available solutions for the power of veto measuring maybe formulated as the following algorithm:

- Re-arrange partition structure including the logic of veto, and
- Apply any power index.

The main problem: what power index is "the best one" is still open. There is a huge of papers in literature on domination of a given index over all or some others, but in no one paper there is a reference to veto itself. Moreover, we think that veto changes axioms being in the background of power indices and we face the problem of how to define power of veto and how to measure this power.

5 Axiomatizing veto

The idea of power index for partition structure with veto strongly depends on kind of veto (Mercik, 2011): the veto of the first degree as the one which cannot be overruled (Security Council of the UN is a good example of such type of veto) and the veto of the second degree as the one which can be overruled, as for the President of the United States or the President of Poland.

In the literature (for example in Kitamura, Inohara, 2009) there is a concept of blockability as ability to block the final result in voting. Let's first check whether veto is equivalent to blocking.

Blockability principle (Kitamura, Inohara (2009)):
Consider a simple game *(N, v)*. For coalitions S and S' $S \gtrsim^b S'$ is defined as: for all $T \in W(v)$, if $T \setminus S' \notin W(v)$ then $T \setminus S \notin W(v)$. $\gtrsim^b$ is called the blockability relation on *(N, v)*.

As we may see the blockability principle is fulfilled only for veto of the first degree. A veto of the second degree may not fulfil this principle. So, blockability principle is stronger than veto: $(N, v, \gtrsim^b) \subseteq (N, v, veto)$, and, blockability may be not equivalent to veto.

In trying to axiomatise an index one looks for “natural” principles that an index should satisfy and then obtain the particular index as the unique solution satisfying these principles if such index exists.

The following axioms are widely accepted:

Axiom 1. (Value-added).

$$\varphi(v)(i_{veto}) \geq \varphi(v)(i) \qquad (4)$$

For veto of the first degree one gets strong inequality $\varphi(v)(i_{veto}) > \varphi(v)(i)$ (some of possible winning coalitions of the others may-be prohibited). What more, $\varphi(v)(i_{veto}) - \varphi(v)(i)$ one may call a net value of veto.

For veto of the second degree, $\varphi(v)(i_{veto}) \geq \varphi(v)(i)$ holds. For example, veto of Polish Senate (if it is a case) can be overruled in all circumstances.

Axiom 2. (Gain-loss: GL axiom).

$$\varphi(v)(i) > \varphi(w)(j) \qquad (5)$$

for some $v, w \in SG$ **and** $i \in N$ **, then there exists** $j \in N$ **such that** $\varphi(v)(j) < \varphi(w)(j)$.

If we introduce veto, the “gain-loss” axiom looks like $\varphi(v)(i_{veto}) > \varphi(w)(j)$ for some $v, w \in SG$ and $i_{veto} \in N$, then there exists $j \in N$ such that $\varphi(v)(j) < \varphi(w)(j)$. We simply assume that right to veto may potentially increase value of power index for a given player. In that case someone else must lose some of its power. Axiom GL is weaker than efficiency and quantitatively less demanding. It specifies neither the identity of *j* that loses power on account of *i*’s gain, nor the extend of *j*’s loss.

Axiom 3 (Efficiency).

$$\sum_{i \in N} \varphi(v)(i) = 1 \qquad (6)$$

for every $v \in SG$ with coalition structure with veto.

For coalition structure with veto $v, w \in SG$ ***define*** $v \wedge w, v \vee w \in SG$ by:

$$(v \vee w)(S) = \max\{v(S), w(S)\},$$

$$(v \wedge w)(S) = \min\{v(S), w(S)\}$$

for all $S \in 2^N$. It is evident that SG is closed under operations $\wedge, \vee$. Thus a coalition is winning in $v \vee w$ if, and only if, it is winning in at least one of *v* or *w*, and it is winning in $v \vee w$ if, and only if, it is winning in both *v* and *w*.

Axiom 4 (Transfer).

$$\varphi(v \vee w) + \varphi(v \wedge w) = \varphi(v) + \varphi(w) \qquad (7)$$

for $v, w \in SG$. This axiom stays same for SG with or without veto.

Axiom 4'. (Transfer – Dubey *et al.* 1981).

Consider two pairs of games *v, v'* and *w, w'* in SG with coalition structure with veto and suppose that the transitions from *v'* to *v* and *w'* to *w* entail adding the same set of winning coalitions, i.e. $v \geq v', w \geq w',$ ***and*** $v - v' = w - w'$. Equivalent transfer axiom: $\varphi(v) - \varphi(v') = \varphi(w) - \varphi(w')$, i.e. that the change in power depends only on the change in the voting game.

Denote by $\Pi(N)$ the set of all permutations of N (i.e., bijections $\pi : N \rightarrow N$). For $\pi \in \Pi(N)$ and a game $v \in SG$, define $\pi v \in SG$ by $(\pi v)(S) = v(\pi(S))$ for all $S \in 2^N$. The game πv is the same as *v* except that players are relabelled according to π.

Axiom 5 (Symmetry).

$$\varphi(\pi v)(i) = \varphi(v)(\pi(i)) \qquad (8)$$

for every $v \in SG$ with coalition structure with veto, every $i \in N$ (including i_{veto}) and every $\pi \in \Pi(N)$.

According to symmetry, if players are relabelled in a game, their power indices will be relabelled accordingly. Thus, irrelevant characteristics of the players, outside of their role in the game *v*, have no influence on the power index.

It seems obvious that introduction of a coalition structure with veto will not demolish this axiom.

Axiom 5' (Equal Treatment).

If $i, j \in N$ are substitute players in the game $v \in SG$ with veto, i.e. for every $S \subset N \setminus \{i, j\}$ $v(S \cup \{i\}) = v(S \cup \{j\})$, then $\varphi(v)(i) = \varphi(v)(j)$.

Axiom 6 (Null player).

If $i \in N$, and *i* is null player in *v*, i.e. $v(S \cup \{i\}) = v(S)$ for every $S \subset N \setminus \{i\}$, then $\varphi(v)(i) = 0$.

The null player cannot be attributed with veto. Otherwise, from "Added value axiom" we get $\varphi(v)(i_{veto}) \geq 0$ what may violate "Null player axiom". To some extend the Supreme Court is the null player with veto attribute. In the example of legislative way in Poland, the Supreme Court doesn't form a coalition with other sides of legislative process (it is independent by definition) but may stop the process if legal contradictories are found.

Axiom 7 (Dummy).

If $v \in SG$, and i is a dummy player in v, i.e. $v(S \cup \{i\}) = v(S) + v(\{i\})$ for every $S \subset N \setminus \{i\}$, then $\varphi(v)(i) = v(\{i\})$.

Dummy axiom implies that $\sum_{i \in N} \varphi(v)(i) = 1$ in every game $v \in SG$ where all players are dummies. The dummy player cannot be attributed with veto. Otherwise, from "Added value axiom" we get $\varphi(v)(i_{veto}) \geq 0$ what may violate "Null Player Axiom" and "Dummy Axiom".

Axiom 8 (Local monotonicity).

LM requires that a voter *i* who controls a larger share of vote cannot have a smaller share of power than a voter *j* with a smaller voting weight. This axiom may not be applied for simple games with veto structure.

A voter *i* is called "at least as desirable as" voter *j* if for any coalition S such that the union of S and $\{j\}$ is winning coalition, the union of S and $\{i\}$ is also winning (LM is an implication of desirability).

Axiom 9 (Desirability with veto).

A voter *i* is called "at least as desirable as" voter *j* if for any coalition S such that the union of S and $\{j\}$ is winning coalition with at least one member with veto power, the union of S and $\{i\}$ is also winning and at least one member has veto power too.

Summing the conclusions from the above analysis of axioms we may say that at least three axioms may not be applied to simple games with coalitional structure with veto, namely: null player axiom, dummy player axiom and local monotonicity axiom. However, the last one can be replaced by the axiom temporarily called "desirability with veto".

In the paper (Einy, Haimanko, 2010) one can find the following two theorems**:**

Theorem 1: There exists one, and only one, power index satisfying Gain-Loss, Transfer, Symmetry and Dummy, and it is Shapley-Shubik power index.

Theorem 2: There exists one, and only one, power index satisfying Gain-Loss, Transfer, Equal Treatment and Dummy, and it is Shapley-Shubik power index.

It is obvious that both above theorems are not valid for simple games with coalitional structure with veto. In consequence, it may exclude Shapley-Shubik power index from the list of potential power indices for such cases, i.e. simple (voting) games where veto is applied. Probably, the same conclusion maybe formulated for Banzhaf power index too. It makes the problem of measuring power for decision making via voting where veto maybe applied still unsolved and using of Johnston power index is on intuitive base only.

6 Conclusions

The analysis of axioms connected with power indices for simple games with coalitional structure with veto leads to the following results: (1) Not all classical axioms maybe assumed for simple games with coalitional structure with veto, (2) Intuitive choice of Johnston power index for simple (voting) games with coalitional structure with veto is still valid, (3) The problem of finding the one and only one power index for simple (voting) games with coalitional structure with veto is still an open case.

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