GIANFRANCO GAMBARELLI

GIOCHI COMPETITIVI E COOPERATIVI

Con un contributo di GUILLERMO OWEN

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G. GAMBARELLI – GIOCHI COMPETITIVI E COOPERATIVI

Games in characteristic function form

N = {1, 2, 3, 4}

Coalitions

(∅) (1) (2) (3) (4)

(1,2) (1,3) (1,4) (2,3) (2,4) (3,4)

(1,2,3) (1,2,4) (1,3,4) (2,3,4)

(1,2,3,4)

Characteristic function: v(S)
v(∅) = 0

Example 1

$v(\emptyset) = 0$

$v(1) = 7$

$v(2) = 3$

$v(3) = 5$

$v(1,2) = 10$

$v(1,3) = 18$

$v(2,3) = 9$

$v(1,2,3) = 100$

Superaddittivity

$v(S) + v(R) \leq v(S \cup R)$

$S \cap R = \emptyset$

$$\begin{cases} \mathbf{v(1) + v(2) \leq v(1,2)} \\ \mathbf{v(1) + v(3) \leq v(1,3)} \\ \mathbf{v(2) + v(3) \leq v(2,3)} \end{cases}$$

Imputations

$v(\emptyset) = 0$

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$v(1,3) = 18$

$v(2,3) = 9$

$v(1,2,3) = 100$

$$\begin{cases} x_1 \geq 7 \\ x_2 \geq 3 \\ x_3 \geq 5 \\ x_1 + x_2 + x_3 = 100 \end{cases}$$

Imputations

$$x = (x_1, \ldots, x_n)$$

$$\begin{cases} x_i \geq \mathrm{v}(i) & \text{INDIVIDUAL RATIONALITY} \\ \sum_{i=1}^{n} x_i = \mathrm{v}(N) & \text{GROUP RATIONALITY} \end{cases}$$

Core (Gillies, 1953)

$v(\emptyset) = 0$

$v(1) = 7$

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$v(1,2,3) = 100$

$$\begin{cases} x_1 \geq 7 \\ x_2 \geq 3 \\ x_3 \geq 5 \\ x_1 + x_2 \geq 10 \\ x_1 + x_3 \geq 18 \\ x_2 + x_3 \geq 9 \\ x_1 + x_2 + x_3 = 100 \end{cases}$$

Core

$$\begin{cases} \sum_{i \in S} x_i \geq v(S) \\ \sum_{i \in N} v_i = v(N) \end{cases}$$

Example - 2

$v(\emptyset) = 0$

$v(1) = 0$

$v(2) = 0$

$v(3) = 0$

$v(1,2) = 0.7$

$v(1,3) = 0.7$

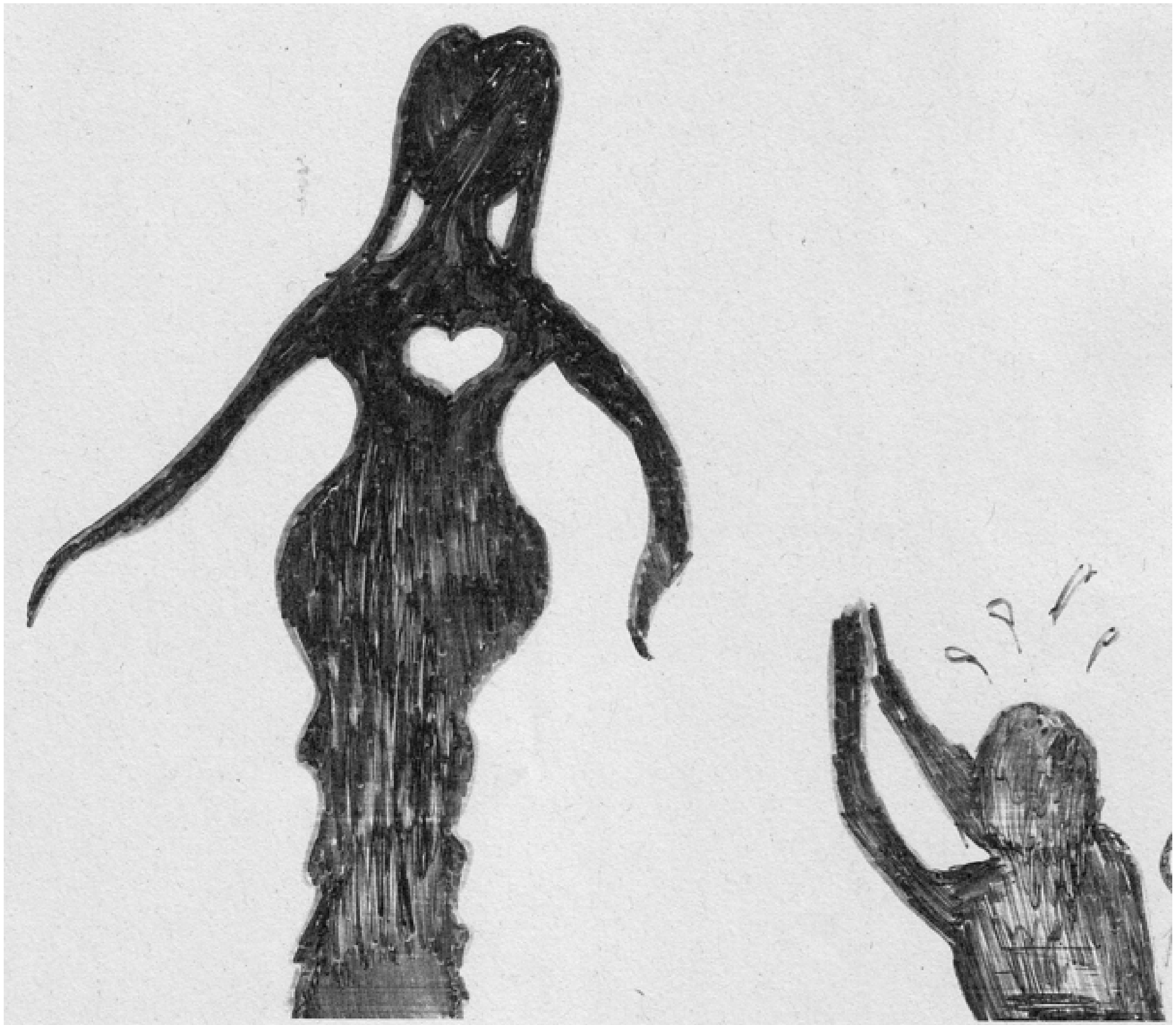
$v(2,3) = 0.7$

$v(1,2,3) = 1$

$$\begin{cases} x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0 \\ x_1 + x_2 \geq 0.7 \\ x_1 + x_3 \geq 0.7 \\ x_2 + x_3 \geq 0.7 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

$$\begin{array}{l} x_1 + x_2 \geq 0.7 \\ x_1 + x_3 \geq 0.7 \\ x_2 + x_3 \geq 0.7 \\ \hline 2(x_1+x_2+x_3) \geq 2.1 \end{array} \Longrightarrow x_1+x_2+x_3 \neq 1$$

empty core



1944	von Neumann – Morgestern STABLE SETS - No existence (Lucas, 1969) - No uniqueness
1953	CORE (as above)

1953	
Gillies	Core
Shapley	A value for TU n-person games

J. F. BANZHAF [1965] & COLEMAN [1971]

$$\beta_i(v) = K \sum_{S \subseteq N} [v(S) - v(S \setminus \{i\})]$$

v(∅) = 0 v(1,2) = 1

v(1) = 1 v(1,3) = 7

v(2) = 0 v(2,3) = 5

v(3) = 5 v(1,2,3) = 18

	Contributions to the coalitions of				
	One player	Two players	Three players	Total	β
1	**1**	(1-0) + (7-5) = **3**	18 – 5 = **13**	**17**	**17·(18/61) ~ 5,01**
2	**0**	(1-1) + (5-5) = **0**	18 – 7 = **11**	**11**	**11·(18/61) ~ 3,25**
3	**5**	(7-1) + (5-0) = **11**	18 – 1 = **17**	**33**	**33·(18/61) ~ 9,74**
				61	**~ 18,00**

L. S. SHAPLEY [1953]

$$\Phi_{i} = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(s-1)!\,(n-s)!}{n!} [v(S) - v(S \setminus \{i\})]$$

	Contributions to the coalitions of			
	One player	Two players	Three players	
1	1	3	13	1·(1 / 3) + 3 ·(1 / 6) + 13·(1 / 3) = 31 / 6
2	0	0	11	0·(1 / 3) + 0 ·(1 / 6) + 11·(1 / 3) = 22 / 6
3	5	11	17	5·(1 / 3) + 11 ·(1 / 6) + 17·(1 / 3) = 55 / 6
Coeff.	1 / 3	1 / 6	1 / 3	Total 18

$$5! = 5 \cdot 4$$

$$1! = 1$$

$$0! = 1$$

$$\frac{(s-1)!\,(n-s)!}{n!}$$

$$\frac{(1-1)!\,(3-1)!}{3!} = \frac{2!}{3!} = \frac{2 \cdot 1}{3 \cdot 2} = \frac{1}{3}$$

$$\Phi$$

$$s = 2$$

$$\frac{(2-1)!\,(3-2)!}{3!} = \frac{1!}{3!} = \frac{1}{3 \cdot 2} = \frac{1}{6}$$

$$s = 3$$

$$\frac{(3-1)!\,(3-3)!}{3!} = \frac{2!}{3!} = \frac{2 \cdot 1}{3 \cdot 2} = \frac{1}{3}$$

The Shapley value (1953)

AXIOMS

SYMMETRY	THE VALUE DOES NOT DEPEND ON THE LABELING
DUMMY PLAYER	IF A PLAYER DOES NOT MAKE ANY CONTRIBUTION: $(\phi_i = v(i))$
ADDITIVITY	$\phi_i (v' + v'') = \phi_i(v') + \phi_i(v'')$

Additivity

$v'(1) = 0$	$v''(1) = 10$	$v^T(1) = 10$
$v'(2) = 1$	$v''(2) = 20$	$v^T(2) = 21$
$v'(1,2) = 2$	$v''(1,2) = 30$	$v^T(1,2) = 30$

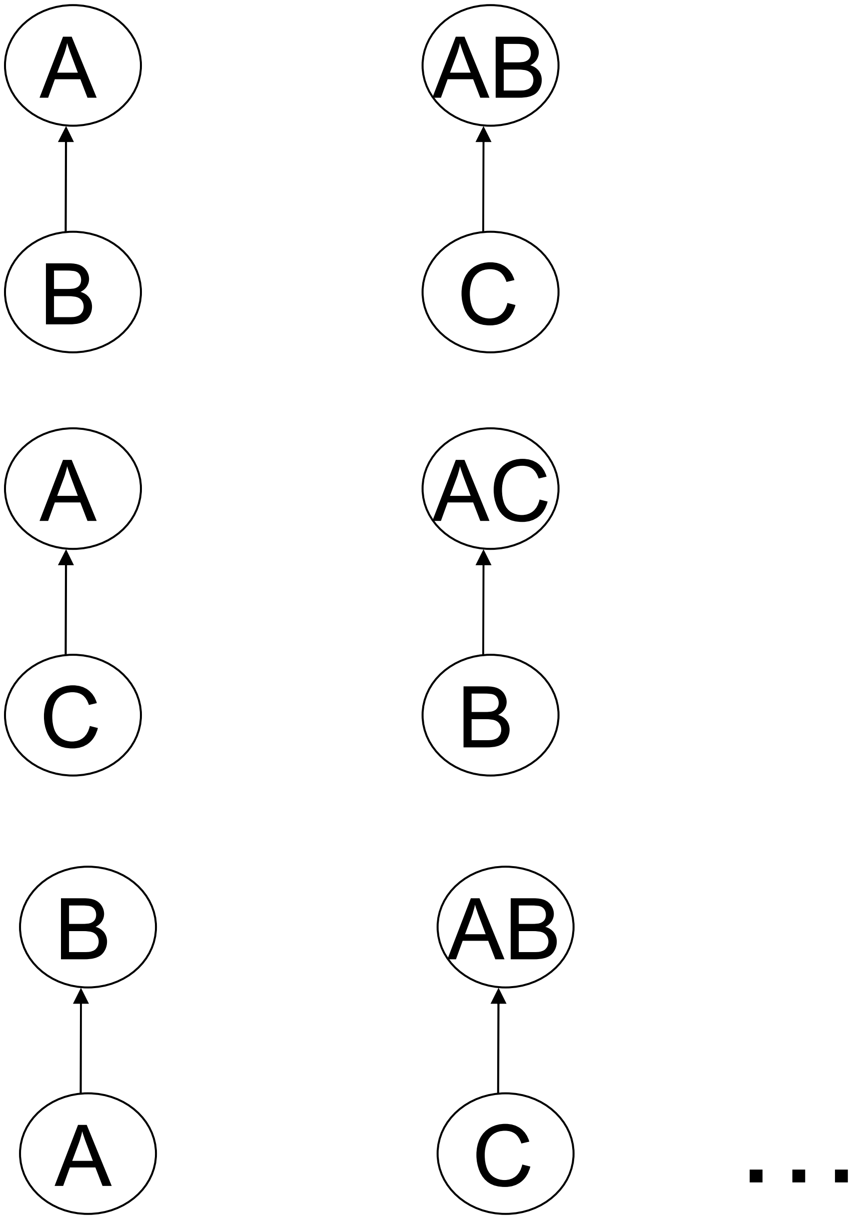
$\Phi'_1 = 0.5$	$\Phi'_1(1) = 10$	$\Phi^T_1(1) = 10.5$
$\Phi'_2 = 1.5$	$\Phi'_2(1) = 20$	$\Phi^T_2(1) = 21.5$

PROPERTIES OF THE SHAPLEY VALUE :

- $\in$ Core for convex games

- Monotonicity

APPLICATION: FORECAST


A
B
AB
C
A
C
AC
B
B
A
AB
C
. . .

The normalized Banzhaf value (1965)

AXIOMS

SYMMETRY THE VALUE DOES NOT DEPEND ON THE LABELING

DUMMY PLAYER IF A PLAYER DOES NOT MAKE ANY CONTRIBUTION: $(\phi_i = v(i))$

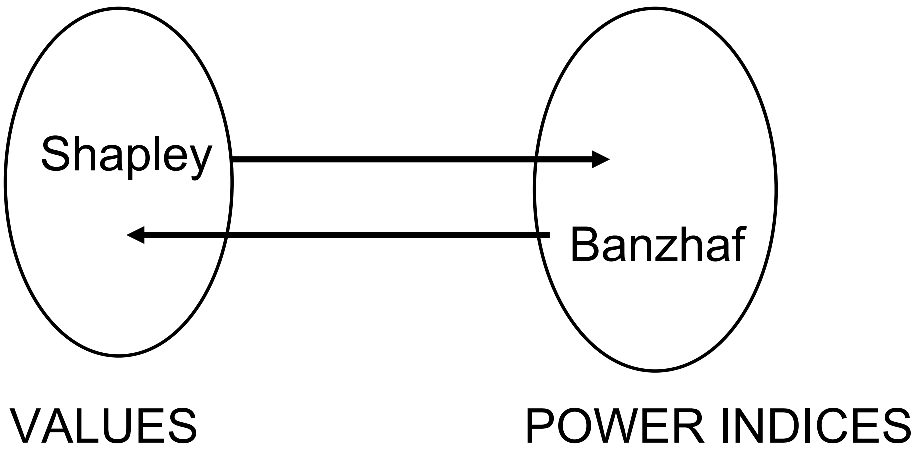
CONCORDANCE WITH THE MARGINAL CONTRIBUTIONS

APPLICATION: LAW

	SHAPLEY	BANZHAF
FORMULA	$\sum_{\substack{S\subseteq N \\ i\in S}} \frac{(s-1)!\,(n-s)!}{n!}$ $[v(S) - v(S\backslash\{i\})]$	$K \sum_{S\subseteq N}$ $[v(S) - v(S\backslash\{i\})]$
STRUCTURE	permutations	combinations
AXIOMS	simmetry dummy player additivity	simmetry dummy player concordance m.c.
PROPERTIES	$\in$core in c.g. monotonicity	- -
APPLICATIONS	forecast	law

Simple games

$$v(S) \in \{0,1\}$$



A general formulation of power indices

$$\phi_i(v) = \frac{1}{a} \sum_{s \in Q} b(s)\ f_i(s) \qquad i = 1, 2, \ldots, n$$

POWER INDICES (FROM VALUES)	ϕ	a	Q	$b(s)$	$f_i(s)$	
SHAPLEY–SHUBIK	σ	1	N	$\frac{(s-1)!\,(n-s)!}{n!}$	$c_i(s)$	$c_i(s)$: number of coalitions (of s members) for which the i-th player is crucial
BANZHAF–COLEMAN	β	$\sum_{j \in N} c_j$	$\{1\}$	1	c_i	
LEMAIRE	L	$2^{(n-1)} - 1$	B	1	$x_i(B_S)$	B = set of bipartitions of N $x_i(B_S)$ = payoff of player i with respect to bipartition B_S
HARSANYI-NASH	η	n	$\{1\}$	1	1	If player i wins alone $\rightarrow$ $\eta_i(v) = 1$ and $\eta_j(v) = 0 \quad \forall j \neq i$
TIJS)	τ	j^v	$\{1\}$	1	1	If player $i \notin J^v$ (set of veto players) $\rightarrow$ $\tau_i(v) = 0$

A general formulation of power indices

$$\phi_i(v) = \frac{1}{a} \sum_{s \in Q} b(s)\, f_i(s) \qquad i = 1, 2, \ldots, n$$

POWER INDICES (DIRECT)	ϕ	a	Q	$b(s)$	$f_i(s)$	
HOLLER	h	$\sum_{j \in N} w_j^*$	$\{1\}$	1	w_i^*	
JOHNSTON	ζ	$\sum_{j \in N} \sum_{S \in C_j} \frac{1}{m_S}$	C_i	1	$\frac{1}{m_S}$	m_S is the number of crucial players of the crucial coalitions If $C_i = \varnothing \rightarrow \zeta_i(v) = 0$
DEEGAN–PACKEL	δ	w^*	N	$\frac{1}{s}$	$w_i^*(s)$	$w_i^*(s)$ Number of **winning coalitions** (of s members) containing the i-th player
SYED	ψ	w	N	$\frac{1}{s}$	$w_i(s)$	$w_i(s)$ Number of **minimal winning coalitions** (of s members) containing the i-th player
CAPLOW	$K(\wp)$	$\sum_{j \in N} \alpha(\wp)_j$	$\{1\}$	1	$\alpha(\wp)_i$	$\alpha(\wp)_i$ *is the number of players that player* i *controls in coalition structure* $\wp$ If $\sum_{j \in N} \alpha(\wp)_j = 0 \rightarrow K(\wp)_i(v) = 0$
GAMSON	$\gamma(\wp)$	$\sum_{j \in S} \omega_j$	$\{1\}$	1	ω_i	ω_i *is the weight of the* i-*th player* If $\sum_{j \in S} \omega_j = 0$ or $S \notin W \rightarrow \gamma(\wp)_i(v) = 0$

COMPARING POWER INDICES

C. BERTINI, J. FREIXAS, G. GAMBARELLI, I. STACH

YES
?
NO

Name	Shapley-Shubik	Banzhaf absolute	Banzhaf relative	Rae	Coleman		Harsanyi -Nash	Deegan-Packel	Johnston	Tijs	PGI	Konig-Brauninger	PHI	Shift
					prevent action	initiate action								
Axiomatization	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	?	YES	YES
Bicameral Meet	NO	YES	YES	NO	YES	YES	YES	NO	NO	YES	YES	YES	YES	YES
Block	YES	YES	NO	YES	YES	YES	YES	NO	NO	YES	NO	YES	?	NO
Dominance	YES	YES	YES	YES	YES	YES	YES	NO	YES	YES	NO	YES	YES	NO
Donation	YES	YES	NO	YES	YES	YES	YES	NO	NO	YES	NO	YES	YES	NO
Efficiency	YES	NO	YES	NO	NO	NO	YES	YES	YES	YES	YES	NO	NO	YES
Null player	YES	YES	YES	NO	YES	YES	NO	YES	YES	YES	YES	NO	NO	YES
Positivity	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES
Simmetry	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES
Transfer	YES	YES	NO	YES	NO	NO	NO	NO	NO	NO	NO	NO	NO	NO
Applications	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES	YES

Weighted majority games

PLAYERS	SEATS	POWER
A	40	1/3
B	30	1/3
C	30	1/3
A	49	1/3
B	49	1/3
C	2	1/3
A	51	1
B	48	0
C	1	0
A	50	3/5
B	30	1/5
C	20	1/5

Banzhaf – Coleman index

$\beta_i = K \sum$ (all crucialities of the i-th player)

PARTY	BEFORE		AFTER	
	SEATS	**BANZHAF**	SEATS	**BANZHAF**
DC	234	35.3	206	42.7
PDS (ex PCI)	177	21.2	107	13.3
PSI	94	21.2	92	13.0
Lomb. League	1	1.3	55	8.4
Com. Ref (ex PCI)	0	-	35	4.6
MSI	35	3.9	34	4.5
PRI	21	6.5	27	3.5
PLI	11	2.0	17	2.3
PSDI	17	2.7	16	2.1
Green	13	2.3	16	2.1
NET	0	-	12	1.6
Pannella (ex. Rad.)	13	2.3	7	0.9
SVP	3	0.5	3	0.5
Other	8+1+1+1	0.8	1+1+1	0.5
TOTAL	**630**	**100**	**630**	100

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	SEATS	BANZHAF	SEATS	BANZHAF
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Shapley – Shubik index

$$\Phi_i = \sum \frac{(s-1)!\,(n-s)!}{n!}$$

↑

extended to all $S \subseteq N$ (s members) for which the i-th player is crucial

PARTY	BEFORE		AFTER	
	SEATS	SHAPLEY	SEATS	SHAPLEY
DC	234	39.3	206	41.6
PDS (ex PCI)	177	22.1	107	15.5
PSI	94	22.1	92	13.8
Lomb. League	1	0.1	55	7.1
Com. Ref (ex PCI)	0	-	35	4.2
MSI	35	5.1	34	4.1
PRI	21	2.8	27	3.4
PLI	11	1.5	17	2.3
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Other	8+1+1+1	1.3	1+1+1	0.6
TOTAL	**630**	**100**	**630**	100

Weighted majority games

w: $(w_1, \ldots, w_n)$

$$\begin{cases} w_h \geq 0 & (h = 1, \ldots, n) \\ \sum_{h=1}^{n} w_h = t \end{cases}$$

q $\left(> \frac{t}{2}\right)$ **majority quota**

$$v(S) = \begin{cases} 1 & \sum_{h \in S} w_h \geq q \quad \textbf{Winning coal.} \\ 0 & \sum_{h \in S} w_h < q \quad \textbf{Losing coal.} \end{cases}$$

Electoral Systems

	Votes	Seats
A	50	5
B	30	3
C	20	2
Totals	100	10

$$s_i = v_i \cdot S / V$$

BUT...

	Votes	Seats
A	50	2.5 ?
B	30	1.5 ?
C	20	1.0 ?
Totals	100	5

CRITERIA OF ROUNDING

- Equal votes → Equal seats
- Monotonicity
  (*more votes → not less seats*)
-Symmetry
- Hare (*roundings*)
- Super-additivity
- Majority (*power indices*)

VOTES	A	B	C	Totals
District .I	50	60	10	120
District II	10	10	60	80
National Totals	60	70	70	200

Hamilton (1)	A	B	C	Totals
I	3	3	0	6
II	1	0	4	5
Totals	4	3	4	11

Breaks

- District I: power index criterion

- Totals: symmetry and monotonicity criteria

Hondt	A	B	C	Totals
I	3	3	0	6
II	0	0	5	5
Totals	3	3	5	11

Breaks:
- District I: power index criterion
- District II: Hare maximum criterion
- Totals: Hare maximum and symmetry criteria

Bal.&Young (1)	A	B	C	Totals
I	3	3	0	6
II	0	1	4	5
Totals	3	4	4	11

Breaks:
- **District I: power index criterion**

OLD METHODS

Use the techinique → Cry on breaks

NEW METHOD

Ranking of criteria:

1)

2)

…

Existence Theorem

Minimax	A	B	C	Totals
I	2	3	1	6
II	1	1	3	5
Totals	3	4	4	11

The minimax apportionment respects both at a local and a national level:

- Symmetry
- Monotonicity
- Hare minimum
- Hare maximum
- Equal seats for equal votes
- Power Indices

THE ADVANTAGES

To the party (or to the coalition) having relative majority

1) the uninominal voting system
2) the majority prize.

To the remaining average and large parties

1) thresholds
2) greatest divisors.

To the smaller parties with peculiar linguistic or ethnical characteristics

1) the respect for such minorities.

To the remaining small parties

1) quota and jump greatest divisors
2) the proportional voting system.

No party
gets any advantage
from the Minimax Method

because this method:

1) respects all the principal equity criteria

2) minimizes distortions as far as possible.

Probably

it will never be adopted