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Power Indices in Politics: Some Results and Open Problems

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Abstract We present an overview of the political applications of power indices, carried out at the University of Bergamo with partners in Europe and United States. New designs, explanations and examples are added so as to better illustrate the results obtained. Additionally, certain open problems are described.

Keywords apportionment, Banzhaf index, electoral system, power index, Shapley-Shubik index, voting, weighted majority game

1. Introduction

This paper presents an overview of the political applications of power indices. The introductory part contains an explanation of the principal game-theoretical instruments that will be used in the sequel. Section 2 gives illustrations of constant power positions. We answer to some questions such as: in which cases do some power indices behave in a similar way even with different numerical values? Section 3 deals with voting shifts between two parties. Section 4 is about the effects of possible changes in electoral rules. Section 5 deals with a new electoral system able to satisfy the most important criteria of fairness. Section 6 considers ideological affinities and distances between parties, as well as multi-cameral systems. Some open prob-

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lems will be illustrated during the course of this paper and will be summed up in the closing section.

1.1 *Historical Background*

The first approach to measuring voting power dates back to the 1780s and was carried out by Luther Martin (see Riker (1986)). The first scientific discussion on the matter was made by Lionel Penrose in (1946). He introduced the concept of 'voting power', a measurement of the ability of a participant of the voting body to influence the outcome. A concept similar to Penrose's was proposed by John F. Banzhaf (1965) and then by James S. Coleman (1971). In 1954, Lloyd S. Shapley and Martin Shubik introduced their power index, which still today is considered by many scholars the best suited describing a large number of application contexts. Since then, numerous other power indices have been proposed, based on various axiomatic grounds and/or bargaining models. For instance, Bertini, Gambarelli and Stach (2008), Caplow (1956), Colomer (1996), Curiel (1987), Deegan and Packel (1978), Gamson (1962), Holler (1978, 1982), Johnston (1978), Mercik (2000), Rae (1969), Riker (1964), Schmeidler (1969), van den Brink and Gilles (1994). Several books have presented the theory of power indices from a variety of viewpoints for example Brams (1975), Brams, Peter and Fishburn (1983), Felsenthal and Machover (1998), Hołubiec and Mercik (1994), Laruelle and Valenciano (2008), Nurmi (1987 and 1999) and Owen (1995). Historical considerations on power indices can be found in Felsenthal and Machover (2005), Gambarelli and Owen (2004) and Meskanen and Nurmi (2008). Some collection of papers should be mentioned that had some impact on the development of power index theory, e.g. Holler (1982) and Holler and Owen (2001). Last but not least it is important to signal the series Oldenbourg's Politics, Philosophy and Economics, edited by Manfred J. Holler and Hannu Nurmi: see i.e. Schofield (2008).

1.2 *Some Preliminary Definitions*

Let $N = \{1,2,\dots,n\}$ be a nonempty finite set. By a game on N we shall mean a real-valued function v whose domain is the set of all subsets of N such that $v(\emptyset) = 0$. We refer to any member of N as a 'player', and to any subset of N as a 'coalition'. A game v is said to be 'simple' if function v assumes values only in the set $\{0,1\}$: $v(S) = 0$ or $v(S) = 1$ for all coalitions $S \subseteq N$. In the first case the coalition is said to be losing; in the second case, winning.

Let us consider an assembly composed of a set $N = \{1, \ldots, n\}$ of members. Each i-th player is given a certain 'weight' w_i (which can represent votes, seats, shares and so on). Let t be the sum of weights of all players in N. Given 'majority quota' q $(> t/2)$ the elements make up a 'weighted majority game', which is usually indicated by the symbol $[q; w_1, \ldots, w_n]$. In this type of game, each coalition S of members of N is called a 'winning coalition' if the sum of weights of its components is equal to or greater than q; otherwise, it is called a 'losing coalition'.

The i-th player is called 'crucial' for the coalition S if S is a winning coalition, but becomes a losing coalition without the contribution of this player. Let denote by $C(i, v)$ the set of all coalitions S for which the i-th player is crucial in the game v (i.e. $v(S) = 1$ and $v(S \backslash \{i\}) = 0$).

A 'power index' of a weighted majority game is a function designed to fix a fair division, or to represent a reasonable a priori expectation of the share of a global prize among the players. More specifically, in the case of weighted voting game $[q; w_1, \ldots, w_n]$ a power index π assigns voting power $(\pi_1, \ldots, \pi_n)$ to the participants of the voting body.

Let v: $[q; w_1, \ldots, w_n]$ be a weighted majority game. We say that power index π is locally monotonic if $w_i > w_j \Rightarrow \pi_i(v) \geq \pi_j(v)$.

Let v: $[q; w_1, \ldots, w_i, \ldots, w_n]$, v': $[q; w'_1, \ldots, w'_i, \ldots, w'_n]$ be two weighted majority games such that $w_i > w'_i$ for one $i \in N$ and $w_j \leq w'_j$ for all $j \neq i$. We say that power index π is globally monotonic if $\pi_i(v) \geq \pi_i(v')$.

Gambarelli in (1983) proposed the following definition of strong monotonicity: $\pi(v)$ is strongly monotonic if, for all i and for all weighted majority games v, v' with $C(i, v) \subset C(i, v')$, $\pi_i(v) < \pi_i(v')$.

1.3 *The Main Power Indices*

Penrose, Banzhaf and Coleman introduced indices that under various aspects may be considered equivalent. The 'normalized Banzhaf index' β represents a summary of such indices. This index assigns each player a quota proportional to the number c_i of coalitions for which he is crucial. For instance, if the weights of a weighted majority game are w_1= 20, w_2= 30 and w_3= 50 and the majority quota q is 51, then the game can be expressed as $[51; 20, 30, 50]$. Player 1 is crucial only for coalition $\{1, 3\}$; player 2 is crucial only for coalition {2, 3}; player 3 is crucial for three coalitions: {1, 3}, {2, 3} and {1, 2, 3}. As the sum of crucialities is 5, the normalized Banzhaf index assigns 1/5 of the power to player 1, 1/5 to player 2 and 3/5 to player 3.

The 'Shapley-Shubik index' Φ (Shapley and Shubik, 1954) is the expression of the Shapley value (Shapley, 1953) for simple games, such as weighted majority games. This index assigns each i-th player an expected payment corresponding to the probability of finding himself in a crucial position, on joining an established coalition. The formula expressing this index necessitates a reference to the concept of factorial number. Given any positive integer k, the product $1 \cdot 2 \cdot \ldots \cdot k$ is defined as 'factorial' of k and is written $k!$; moreover, the convention $0! = 1$ is used. Indicated as $P(s, n) = (s\text{-}1)!(n-s)!/n!$, the Shapley-Shubik index of every i-th player is given by the sum of all $P(s, n)$ extended to all the coalitions of s members for which the i-th player is crucial. In the above example it is $n = 3$, $0! = 1! = 1$, $2! = 2$, $3! = 6$. As the first player is crucial only for one 2-member coalition, his index is $P(2, 3) = (2\text{-}1)!(3\text{-}2)!/3! = 1/6$; the same is true for the second player. The Shapley-Shubik index of the third player is $P(1, 3) + P(2, 3) + P(3, 3) = 1/6 + 1/6 + 1/3 = 2/3$.

A difference between the normalized Banzhaf index and the Shapley-Shubik index lies in the bargaining model: the former does not take into account the order in which the winning coalition is formed, while the latter does. It is important to note that the first index is not globally monotonic in the sense that if a party gains seats from another party, its normalized Banzhaf index *could* decrease (see example presented in Table 1). On the other hand, the Shapley-Shubik index satisfies this monotonicity property. Both indices however satisfy local monotonicity: players with higher weights have at least as much voting power than players with lower weights. For some in depth consideration on this issue see for example Turnovec (1998), Holler and Napel (2004).

Table 1 — Powers of members in two 5-members assemblies with the majority quota q= 69.

Party	Assembly 1			Assembly 2		
	Seats	Normalized Banzhaf power index	Shapley power index	Seats	Normalized Banzhaf power index	Shapley power index
1	38	$\frac{1}{2}$	$\frac{1}{2}$	46	$\frac{9}{19}$	$\frac{55}{100}$
2	38	$\frac{1}{2}$	$\frac{1}{2}$	30	$\frac{7}{19}$	$\frac{30}{100}$
3	8	0	0	8	$\frac{1}{19}$	$\frac{5}{100}$
4	8	0	0	8	$\frac{1}{19}$	$\frac{5}{100}$
5	8	0	0	8	$\frac{1}{19}$	$\frac{5}{100}$
Totals	100	1	1	100	1	1

With regard to applications, the normalized Banzhaf index is considered the best suited to inclusion in normative models, thanks to its direct proportionality to the number of crucialities. On the other hand, the Shapley-

Shubik index is best suited to forecasting models, owing to its properties of monotonicity and stability (belonging to the core in convex games). As we have seen, other power indices have been built for other applications. The nucleolus (Schmeidler, 1969) and the Public Good Index (Holler, 1978 and 1982) are of particular importance in social contexts.

2. Constant Power Positions

In this section, using a real-world situation as a starting point, we shall focus attention on the existence of constant power positions in weighted majority games.

2.1 *A First Application*

Table 2 shows the powers of the parties after the Italian election of 2008, according to the normalized Banzhaf and Shapley-Shubik indices. Before examining the numerical results, some observations are required on the reliability of the models in question.

Firstly, some coalitions which are possible in theory are not possible in practice. For instance, those between the extreme right and extreme left parties. For the correct application of an index, it seems necessary to discard from the cruciality calculations for each player, all the coalitions that are unfeasible. However, there are decision situations (e.g. referendum or presidential elections) where numerical strength is more important than political closeness; in these cases the original models applies.

It is interesting to note from Table 2 that, as a result of the election of 2008 LNP augmented seats (from 23 to 60), but decreased its decision power in both indices. Basically, this means that the different distribution of seats made LNP crucial for a smaller number of coalitions. Further, a comparison between the LNP and MISTO parties shows that while the former party had almost triple as many seats as the latter, it has the same power (despite the fact that the numerical values differ depending on which index is used). This poses some interesting questions: in which cases do the normalized Banzhaf index and the Shapley-Shubik index, and possibly other indices too, behave in a similar way even with different numerical values? How can these properties be used in applications?

2.2 *The Case of Two Parties*

Using the power indices as a starting point, it is possible to build simulation models of a party's power variations following a modification in seat dis-

Table 2 — Normalized Banzhaf and Shapley-Shubik indices in Italian parliaments.

Party	XV legislation (2006)			Party	XVI legislation (2008)		
	Seats	Banzhaf	Shapley-Shubik		Seats	Banzhaf	Shapley-Shubik
PD	218	24.55	34.05	PD	218	43.750	46.67
FI	134	24.55	34.05	PDL	272	43.750	46.67
AN	72	14.55	9.29	-	-	-	-
RC	41	7.27	4.29	-	-	-	-
UDC	39	7.27	4.29	UDC	34	3.125	1.67
MISTO	83	14.55	9.29	MISTO	19	3.125	1.67
LNP	23	3.64	2.38	LNP	60	3.125	1.67
IDV	20	3.64	2.38	IDV	27	3.125	1.67
TOTALS	630	100.00	100.00	TOTALS	630	100.000	100.00

tribution with particular regard to seat shifts between one party and another and the inclusion of new voters who modify seat redistribution. (The latter case will be examined in the next section.) Suppose the seats of a Parliament are initially distributed between only two parties in the proportion of 75 and 25. In this case, for all decisions that require a majority of 51%, the first party has the majority and therefore the power index is (1, 0), which is to say 1 for the first and 0 for second party. If the division of seats is 10 and 90, the index is (0, 1). If the first player's seats are 50.5% and the second player's seats are 49.5%, the index is (1/2, 1/2).

In general, the set of points on a Cartesian plane, which represents all possible seat distributions, is the oblique segment in Figure 1, where w_1 and w_2 are the weights of the two parties (with the conventions $w_1 \geq 0$, $w_2 \geq 0$ and $w_1 + w_2 = 100$). It should be noted that at all points having abscissa greater than or equal to 51, the power index is (1, 0); at all points having abscissa less than or equal to 49, the power index is (0, 1), and at all other points it is (1/2, 1/2).

Suppose that, starting from position (75, 25), the first party yields seats to the second one (cf. Figure 2). As long as the point lies in the lower right segment, the power remains unchanged and therefore the decrease in power of the first party is zero. If the point reaches the central segment, then the decrease of power is 1/2, while if the point passes it, then the decrease of power is 1.

Fig. 1 — All possible distributions of seat between two parties.

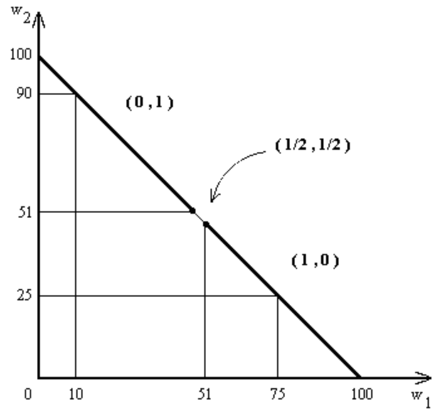
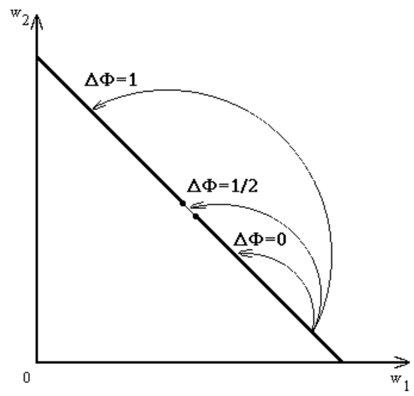


Fig. 2 — Power variations following a modification in seat distribution between two parties.



2.3 *The Case of Three Parties*

Figure 3 shows a three-player game where the vector $w = (w_1, w_2, w_3)$ represents the non-negative weights (seats) of the parties, under the constraints $t = w_1 + w_2 + w_3 = 100$. Owing to the above constraints, the vector w lies within the triangle having vertices (100, 0, 0), (0, 100, 0) and (0, 0, 100). For the sake of simplicity, we will consider the game with simple majority.

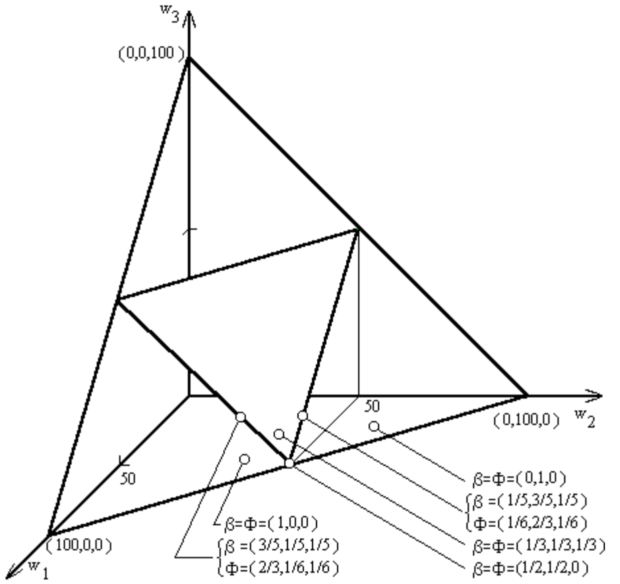
Let us consider the parallel plane to the first and third axis, passing through point (0, 50, 0). At all points to the right of this plane (i.e. where $w_2 > 50$) the second player has a majority, and therefore his power is 1 in these points. Consider the smaller triangle having vertices (0, 100, 0), (50, 50, 0), and (0, 50, 50). The power index is (0, 1, 0) at all points of this triangle (with the exception of the segment joining the last two vertices). Analogously, the index is (1, 0, 0) at all points on the triangle having vertices (100, 0, 0), (50, 50, 0), and (50, 0, 50). It is also possible to verify that the index is (1/3, 1/3, 1/3) at all points on the central triangle having vertices (50, 0, 50), (0, 50, 50), (50, 50, 0) (borders excluded). Regarding the borders, at the internal points of each segment the values differ depending on the chosen index. For instance, for the border which joins the vertices (50, 50, 0) and (0, 50, 50), the normalized Banzhaf index is (1/5, 3/5, 1/5), while the Shapley-Shubik index is (1/6, 2/3, 1/6); the same (with suitable permutations) applies for the others. Finally, on the vertices of the small central triangle, the index is (1/2, 1/2, 0), (1/2, 0, 1/2) and (0, 1/2, 1/2).

Also note that, when the non-limit case is considered, the diagram would show subdivisions of the large triangle not only as small triangles, but also as trapezia. (There are two types of diagram, depending on whether q is greater or less than $2t/3$.) Each of these polygons has the following property: in all points the game is constant, as regards the coalitions for which each player is crucial. Now it is possible to give a intuitive answer to one of the questions posed at the end of the preceding section: Do the Shapley-Shubik index, the normalized Banzhaf index and possibly other indices behave in a similar way, even though they have different numerical values?

2.4 *The Case of n Parties*

'Mathematical' readers can envisage a generalization of what has been examined so far, when applied to games with n players. The triangle in Figure 3 becomes a simplex of the Euclidean n-dimensional space, having vertices in all points so that one of the components is the total sum of weights t and all other components are zero (in our case t= 100). This simplex is subdivided into convex polyhedra by hyperplanes parallel to the main axis and

Fig. 3 — Power variations following a modification in seat distribution between three parties.



where the distance from these is q and $t - q$. In each of these polyhedra, the game is constant. Therefore, once a power index has been chosen to represent the real situation being studied, at all points on each polyhedron the index remains unchanged.

For further information and theorems on the matter, see Gambarelli (1983). Other common properties regarding various power indices may be found, for instance, in Freixas and Gambarelli (1997).

2.5 *The Powers of Subsets*

Parties are often made up of various tendencies, within which there can be even further diversification. In order to quantify the power of each of these subgroups with regard to consensus, the model proposed by Gambarelli and Owen (1994) may be applied. This model relates to the indirect control of corporations, that is a shareholder owns shares of a firm, this firm owns shares of another firm and so on. The model transforms this set of games into a unique game.

3. Vote Shifts Between Two Parties

Another model was studied to predict changes in power relationships that follow from a shift by a subset of the electorate from one party to another (see Gambarelli (1983)).

3.1 *An Example*

Let us assume that the initial distribution of seats among parties A, B and C in a 3-person weighted majority game is (51, 40, 9) (see Table 3). Given simply majority voting ($q \geq 51$) a transfer of seats between B and C will not change the situation, as A will remain the majority party. However, let us now analyse what happens if seats are exchanged between A and C. If C receives one seat from A, the distribution becomes (50, 40, 10) and the power distribution (according to the Shapley-Shubik index) becomes (2/3, 1/6, 1/6). If C receives 2 seats from A, the distribution of seats becomes (49, 40, 11) and the power distribution is (1/3, 1/3, 1/3). The division of power remains the same even if C obtains 40 seats from A, as in this case seat distribution becomes (11, 40, 49) and each player is in the same position as the others. The situation changes only if C receives 41 seats from A: in this case the distribution becomes (10, 40, 50) and the power of C increases to 2/3. With one more seat, C acquires the majority and its power increases to 100%.

Table 3 shows that the power of C is a monotonic step function of the number of seats acquired by A. The critical stocks which allow C to pass from one position of power to another are 9, 10, 11, 50 and 51. In Gambarelli (1983) it was proved that, however a power index is defined (provided that it is strongly monotonic), the sequence of critical stocks corresponding to seats transferred between two players i and j is always the same. The formulae generating these critical stocks d_s in a parliament with n parties are given in this article.

Table 3 — Exchange of seats between two players (according to the Shapley-Shubik index).

Party	Number of seats C receives from A	Resulting distribution of seats	Resulting distribution of power	Power increase of *C*
A		51	1	
B		40	0	
C	0	9	0	0
A		50	2/3	
B		40	1/6	
C	1	10	1/6	1/6
A		49	1/3	
B		40	1/3	
C	2	11	1/3	1/3
A		10	1/6	
B		40	1/6	
C	41	50	2/3	2/3
A		9	0	
B		40	0	
C	42	51	1	1

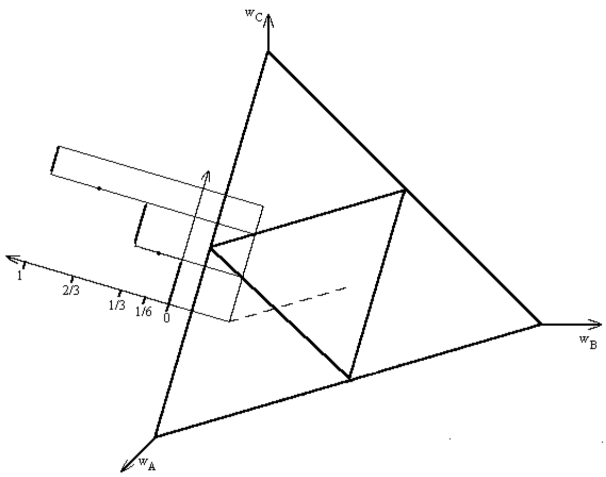
Synthesis

Number of seats (C receives from A)	Resulting increment of power
1	+ 16.7 %
from 2 to 40	+ 33.3 %
41	+ 66.7 %
from 42 to 51	+100.0 %

3.2 *The Most Dangerous Partner*

Figure 4 shows the movement of the seat vector while seats are being transferred between A and C as in Table 3 the number of seats owned by B remains unchanged, the relative component is constant and therefore the point, representing resulting vector of distribution of seats, moves within a plane parallel to the w_A w_C plane. The resulting segment meets the borders of the small triangles at two points, determining the step function (marked at the top right of the diagram) which shows the power increase of C depending on the number of seats received from A. From Figure 4 we can see how the same number of seats transferred can give different results, in terms of power indices, depending on the party that acquires seats. If party A loses 40 seats to C instead of to B, A's power will go from 1 to 1/3 (cf. broken segment), instead of from 1 to 0. It is therefore important to know not only the discontinuity points of the step function (i.e. the critical blocks of seats which enable a player to move from one constant power area to another), but also the most 'dangerous partner' in seat transfer. Results for the general case were presented in (Gambarelli and Szegö, 1982). It should be remembered that the monotonicity mentioned in section 1.3 holds for the Shapley-Shubik index and is not satisfied by the normalized Banzhaf index.

Fig. 4 — Exchange of seats between two parties and following resulting distribution of power.



3.3 *Algorithms*

Many open problems are to be found in the search for properties common to various power indices, especially with reference to alternative vote shifts. On the basis of considerations regarding the geometrical properties of the Shapley value, an algorithm was used for calculating this value in super-additive games (see Gambarelli, 1980). The algorithm was generalized in

Gambarelli (1990) for subadditive games. The algorithm is linear in the number of significant coalitions and uses a theorem of early stop, based on reaching the desired degree of precision. In majority games having a low total sum of weights, the Shapley value (which assumes in these cases the role of the Shapley-Shubik index) can be better calculated using the algorithm proposed by Mann and Shapley (1962). The generation of the 'power' function relative to share exchanges between parties necessitates the repeated usage of Mann and Shapley's algorithm in each of the constant power regions. A subsequent algorithm by Arcaini and Gambarelli (1986) enables further savings in calculation, as it directly generates the increase in the index starting from each point of discontinuity, taking into account the information that was used to calculate the preceding value.

A similar technique was applied in Gambarelli (1996) to generate the power function in the case of the normalized Banzhaf index. This algorithm also provides a direct method of calculating this index. This method turned out to be faster than the one used previously (i.e. the one suggested by Banzhaf (1965)). Certain improvements in the calculation of the normalized Banzhaf index have been proposed subsequently (see for instance Algaba, et al. 2003), but the possibility of further savings in time remains open, especially in the case of seat shifts.

4. Effect of Changes in Electoral Law

4.1 *Entries and Thresholds*

A model of shifts in votes (and corresponding seats) between a player and an ocean of small players was studied in Gambarelli (1983). This article gives formulae for describing index variations following the introduction of electoral laws that extend the vote to new categories of electorate (for example, immigrants, emigrants, young people, etc.). The algorithms mentioned in section 3.3 may be used for these calculations.

A number of regulations fix ceilings for a party's entry into Parliament. Since each ceiling threshold has different effects on the power of parties, it follows that the application of power indices to this problem allows for an evaluation of the ceiling best suited to each party. Table 4 gives an example of power allocation according to the ceiling considered and the Banzhaf power index. The above-mentioned algorithms allow tables such as this to be calculated without losing too much time.

Table 4 — Power in the case of small party thresholds calculated by the Banzhaf power index.

		Power in the case of thresholds of							
Party	% votes	0 %	1 %	2 %	3 %	4 %	5 %	6-8 %	9%
A	37.39	32.41	35.59	37.04	38.46	41.67	44.44	50.00	100.00
B	33.17	19.35	22.07	22.22	23.08	25.00	33.33	50.00	0.00
C	8.30	15.56	16.67	16.67	15.38	16.67	11.11	0.00	–
D	5.62	8.57	9.46	9.26	11.54	8.33	11.11	–	
E	4.37	6.79	7.66	7.41	7.69	8.33	–		
F	3.08	4.90	4.95	5.56	3.85	–			
G	2.43	4.13	2.25	1.85	–				
H	1.13	1.65	1.35	–					
I	0.98	1.44	–						
J	0.81	1.20	–						
K	0.57	0.84	–						
L	0.46	0.67	–						
M	0.41	0.60	–						
N	0.37	0.56	–						
O	0.33	0.48	–						
P	0.30	0.45	–						
R	0.28	0.41	–						
TOTALS	100	100	100	100	100	100	100	100	100

4.2 *Apportionment Strategies for the European Parliament*

The apportionment of seats to incoming members to the European parliament has always been a source of discussion. From among the large amount of studies on the issue we may mention Affuso and Brams (1985), Bilbao (2000 and 2001), Felsenthal and Machover (2001, 2003), Helin, Nurmi and Raunio (2007), Holler and Widgrén (1999), Laruelle and Widgrén (1998), Meskanen and Nurmi (1999), Nurmi (2000), Turnovec (1996) and Widgrén (1994).

In general, in the past the trend has been to take into account the size of population. An apportionment method was proposed in Bertini, et al. (2005). It consisted in restructuring the distribution of seats for all countries using a formula which takes into account both populations and Gross Domestic Products (GDP). For instance, let populations and GDP percentages of the i-th country be shown by P_i and G_i. Suppose that we decide to weight the population with 30% and GDP with 70%. In this case the seat percentages S_i of the i-th country would be S_i= 0.3 P_i + 0.7 G_i. Generally speaking, if a is the weight we assign to the population ($0 \leq a \leq 1$), the resulting seat percentages are $S_i = aP_i + (1\text{-}a)G_i$. The value of a characterizes seat distribution. Accordingly, the parameter a can be expected to be a talking point between countries with strong economies and those with weak economies. From an initial examination, it would seem in the interests of countries with higher GDP percentages than their population percentages (Denmark, Finland etc.) to have lower values of a (possibly 0). Vice versa, for countries with lower GDP percentages (Poland, Romania etc.) it would seem in their interests to have high values of a (possibly 1). However, this rule does not always apply because the power of a country is not proportional to the number of seats when we take in consideration the capacity to form a winning coalition. In Bertini et al. (2005) the evaluation of optimum value of a for each country was done taking into account the Banzhaf index and varying a with 0.1 steps. It is possible also to define 'optimum weight interval of the i-th country' as the variability interval a_i, which guarantees this country the maximum power index, and with equal power index, the maximum number of seats. In considering article were evaluated optimal intervals of a in case the European Parliament consists of only 3 countries. Further studies on the behaviour of optimum weight intervals could be carried out applying the others power indices like those of Shapley-Shubik and Holler. Summarize, with such a method it is possible to study a strategy of optimization for each single country based on the application of different power indices.

5. A New Electoral System that Respects All the Main Criteria of Fairness

The general problem of seat apportionment in electoral systems is quite complex, since no apportionment method exists which is successful in verifying all the principal fairness criteria which have been introduced. Starting from results obtained by Balinski and Demange (1989) and Balinski and Young (1982), Gambarelli (1999) proposed an apportionment technique that is custom made for each case, given that it respects rounding up and down and satisfies other criteria in order of preference. In a more recent work, a generalization of this method has been proposed, in order to extend it to multi-district election case, where criteria should be respected both at global as well as local levels (see Gambarelli and Palestini (2007)).

5.1 *The Apportionment Problem*

The apportionment problem consists of assigning seats to political parties in proportions that best reflect the number of votes obtained, or districts in proportion to population, and so forth. Mathematically the problem consists of transforming an ordered set of non-negative real numbers (votes) into integers (seats) that respect certain equity criteria, all of which appear reasonable. The problem is that not all criteria can be respected simultaneously, because certain criteria contradict each other. Solutions must be therefore found which both respect the most important criteria and minimize distortions. For example, consider a political system composed of three parties. These parties respectively gain 11, 10 and 2 votes in an election. If there are three seats to be assigned, this would mean, in theory, giving $3 \cdot 11/23 \cong 1.43$ seats to the first, $3 \cdot 10/23 \cong 1.30$ to the second and $3 \cdot 2/23 \cong 0.26$ to the third (see the second column of Table 5).

Table 5 — Apportionment following the Proportional System.

VOTES	Quotients	Integer parts	Residual seats	SEATS TOTALS
11	1.43	1	1	2
10	1.30	1	0	1
2	0.26	0	0	0
23	3.00	2	1	3

To make the seats whole, a process of rounding off is necessary, bearing in mind reasonable equity criteria: for example, a major number of votes should not correspond to a smaller proportion of seats ('monotonicity'), an equal number of votes should correspond to an equal proportion of seats and, in general, seat apportionment should not depend upon the order in which parties are considered ('symmetry'). Moreover, an important criterion is that the number of seats assigned to each party should not be lower than the exact relative quotient rounded down ('Hare Minimum'), neither should it be higher than the exact relative quotient rounded up ('Hare Maximum'). 'Superadditivity' dictates another important criterion, which will now be described. If a rounding-off method assigns a certain number of seats to two parties, the same method should assign a number of seats not inferior to the total sum, to the 2-parties-union (obtained via a hypothetical coalition of the two parties).

Another criterion, introduced in Gambarelli and Hołubiec (1990), will be illustrated in Section 5.2.

The simultaneous application of these criteria, which at first sight seem so indispensable, may, however, prove to be unfeasible. For example, if there are two parties and they receive an equal number of votes, it will be impossible to apportion an odd number of seats without violating the criterion of 'equal seats for equal votes'. Generally speaking, it has been shown that any given method of rounding-off which respects symmetry and monotonicity, cannot simultaneously respect Hare Maximum and superadditivity.

5.2 *Classical Methods of Apportionment*

A brief presentation of the better-known rounding-off methods will now be given and applied to the numerical example in Table 5 (The others are, for the most part, variations of these. For further information see also Nurmi (1987).)

Using the *Proportional System* (or *Hamilton's Method*), each party is initially assigned the whole part of theoretical seats (in this case 1, 1, 0). The remaining seats (in this case 1) are apportioned to the parties with the highest fractional share (in this case, the first party, whose fractional share is 0.43). In the example, the resulting apportionment is therefore 2, 1, 0.

Using the *Method of Greatest Divisors* (or the *d'Hondt Method*), the procedure is as follows: votes received by the first party are divided by 1, then by 2, then by 3, etc., for as long as the procedure is necessary. Similar divisions are then made for the votes received by the other parties. The highest quotients (as many as there are seats to apportion) are then taken into consideration and a seat is assigned to each of the parties on the basis of these quotients.

Table 6 — Apportionment following the Method of Greatest Divisors.

VOTES	:1	:2	:3	...
11	$\underline{11}$	$\underline{5.5}$	$3.\bar{6}$	...
10	$\underline{10}$	5	$3.\bar{3}$	...
2	2	1	$0.\bar{6}$	...

In the case of our example (see Table 6), the highest quotients are, in decreasing order, 11, 10 and 5.5; of these, two correspond to the first party (11 and 5.5) and one to the second party (10). Therefore two seats are assigned to the first party and one to the second. In this example, such an apportionment corresponds to the *Proportional System*, though generally the results are different. The *Method of Greatest Divisors* can lead to apportionments that fail to respect Hare Maximum. For this reason Balinski and Young introduced, in 1975, the *Method of Greatest Divisors with Quota*, in which no seat is assigned if the consequent number of seat doesn't respect rounding up. However, for the example being considered, this method is consistent with Table 5.

5.3 *The Criterion of Power Indices*

In order to present the criterion of power indices, introduced in Gambarelli and Hołubiec (1990), we should turn our attention to the example given in Table 6, noting that a percentage apportionment of votes gives, according to the more common power indices, the same index value for all three parties. On the other hand, the apportionment of seats resulting from Tables 5 and 6 gives an absolute majority to the first party and an allocation of power (1, 0, 0). This result (illustrated in Table 7) demonstrates the extent to which apportionment obtained via classical methods is essentially undemocratic. In fact, such methods distort coalition power of the parties, undermining the majority principle fundamental to any democracy.

Figure 5 illustrates an explanation for this phenomenon. The passage from votes to seats implies, in fact, the translation of a point on the vote triangle (the larger in the diagram, with vertices on three axes) to a point on the seat triangle (the smaller of the two).

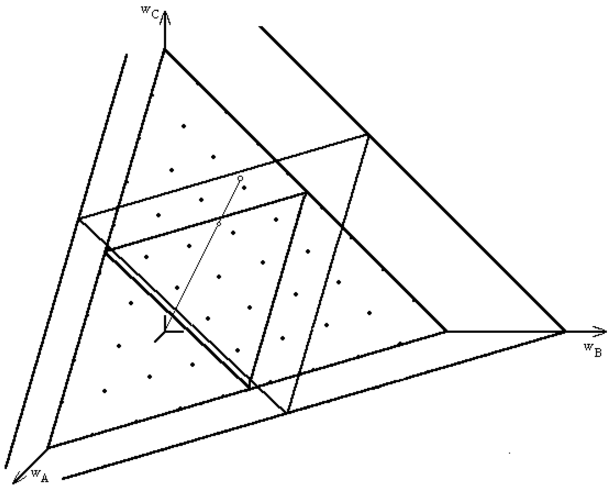
Table 7 — An example of coalition power distortion brought about by the most well-known electoral systems (PS = Proportional System; GD = Greatest Divisors).

VOTES			SEATS	
n.	%	power	PS and GD	power
11	47.83	1/3	2	1
10	43.48	1/3	1	0
2	8.70	1/3	0	0
23	100	1	3	1

However, not all points on the small triangle are acceptable: in fact, the destination point must have integer coordinates. Essentially, we are dealing with a subset of this triangle, made up of the single points indicated in the diagram. By joining the votes vector to the origin a segment is obtained, whose intersection with the small triangle gives an apportionment of seats according to exact quotients.

If an intersection falls on a integer coordinate, then perfect apportionment results. Otherwise, in terms of admissible points, those ‘nearest’ the intersection must be evaluated, according to a suitable metric. Possibly, the point obtained in this way might belong to a constant power region that is different to the corresponding power region in the original triangle. In this case, differences in power indices between votes and seats exist. The criterion of rounding-off consists of respecting, as far as possible, allocations given by the power indices that correspond to votes.

Fig. 5 — Allocations of seats in proportion to the votes.



5.4 *The Method of Minimax*

The classic rounding-off methods have this approach: ‘we apply a technique, then we see which criteria it violated, and we cry over it’. (In the case of greatest divisors with quota only one violation is solved.) The method of minimax (Gambarelli, 1999) overturns this approach by proceeding as follows. Taking into account that all the equity criteria cannot be respected totally, the damage has to be limited as much as possible. Therefore, we base the list (in order of preference) on the criteria we think are the most important. We build the set of seat distributions that satisfy the first criterion. Then we isolate the subset of seats that also satisfy the second criterion (if not empty) and we proceed until we obtain a final set of apportionments. As we will see, this final set is not empty. If during the restrictions we reach a single apportionment, then we stop the calculation. If at the end several apportionments still remain, we proceed with external methods in order to obtain a unique solution (candidate age or sex, type, etc.).

The first criterion to use is a combined respect for monotonicity, Hare Minimum and Hare Maximum. It can be demonstrated that such a criterion generates in each case a non-empty set of apportionments.

As a second criterion, the minimization of the maximum damage is adopted, according to Schmeidler (1969) nucleouls. This minimization is subsequently applied to the percentage apportionment of votes and to one of the power indices (or vice-versa, if the latter is considered a priority). This procedure implies respect for 'equal seats for equal votes' and, in general, symmetry, in all cases where this is possible. These operations reduce the possible set of apportionments, without making it empty.

Gambarelli and Palestini (2007) applied the minimax apportionment method to a multi-district election case, by fixing an analogous sequence of criteria to be respected on both global and local levels, in order to apportion seats in the fairest way and minimize distortions.

6. Affinity and Ideological Distance

The models presented so far have not taken affinity or aversion between parties into account. This makes sense in the case of electoral laws (which must clearly remain apart from such considerations) and for those types of voting in which numbers may be more important than ideological proximity. On the other hand, when affinity and aversion are to be taken into account, more specific models need to be developed. There is a vast quantity of literature available on coalition formation, including Aumann and Dréze (1974), Caplow (1956, 1959 and 1968), Gamson (1961 and 1962), Holler (1982) and Gambarelli (2007). Particularly, Myerson (1977) is devoted to games with restricted communication among the players, modelized by an undirected graph. Owen (1977 and 1982) is referred to games with a priori unions modelized by a partition of the set of players. Recent publication on this issue is Alonso-Meijde et al. (2009).

A second problem lies with multicameral Parliaments. Most parliaments are based on a bicameral system where bills have to be approved by both chambers. The parties' power indices are affected by it. A political party, for a given coalition, can be crucial in one chamber but not in the other. The problem can be solved by building a unified game, related to two or more chambers, for which joint power indices can be calculated. An interesting result is quoted in Taylor and Zwicker (1993): for every assembly it is always possible to find k assemblies so that a coalition is winning in the original assembly if and only if it is winning in all k assemblies. Further contributions have been given, for instance, by Bilbao etc. (2002), by Algaba etc. (2003) and by Turnovec (1992a, 1992b and 1992c).

In the above works multi-cameral games and coalition cohesion have been examined separately. Gambarelli and Uristani (2009) built a model that takes into consideration both problems. The model is applied to those European Union members countries whose parliaments are currently bicameral, where both chambers have a political composition and each chamber can reject a law approved by the other. These countries are Belgium, the Czech Republic, France, Italy, the Netherlands, Poland and Romania. This model also applies to Germany for a subset of lands. The model is even applied globally to the European Union. This model could subsequently be applied to other countries outside Europe.

7. Open Problems

As has been seen at the end of the various sections, there are still many open problems in the research presented here. They may be summed up as follows:

- properties common to various power indices, especially with reference to alternative vote reallocations (section 3.2);
- an improvement in algorithms, especially in the case of seat shifts (section 3.3);
- further simulations related to thresholds (section 4.1);
- applications of the method of minimax (section 5.4);
- applications of the multicameral cohesion games model to countries outside Europe (section 6).

The authors would be pleased to supply material and suggestions to anyone wishing to undertake such work.

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